Unique Continuation and Inverse Problems

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1 Estimates for a $\bar{\partial}$-problem

Sections 2, 3 up to Lemma 3.7 from [1]

A summary written by Gianmarco Brocchi
after Adrian Nachman, Idan Regev, and Daniel Tataru [1]

Abstract

We study solvability of the inhomogeneous problem $L_q u = f$, where $L_q u := \bar{\partial} u + q\bar{u}$, and $q \in L^2$. The authors prove new pointwise bounds for fractional integrals and pseudo-differential operators with non-smooth symbols, as well as new estimates for pointwise multiplier in negative Besov spaces.

1.1 Introduction

Consider the problem

$$\bar{\partial} u + q\bar{u} = f. \tag{1}$$

Indicate with L_q the operator $\bar{\partial} + q\bar{\cdot}$. We want to study the inverse operator L_q^{-1}, particularly the dependence on q of the operator norm $\|L_q^{-1}\|$.

Let $s \in [0,1)$. The operator $\bar{\partial}\colon \dot{H}^s(\mathbb{C}) \to \dot{H}^{s-1}(\mathbb{C})$, as well as the multiplication by q for $q \in L^{1/s}(\mathbb{C})$. When $s = \frac{1}{2}$, $q \in L^2$ and we have

$$L_q\colon \dot{H}^{\frac{1}{2}}(\mathbb{C}) \to \dot{H}^{-\frac{1}{2}}(\mathbb{C}).$$

Our aim is to prove the following theorem.

Theorem 1. *Given $q \in L^2$, for every $f \in \dot{H}^{-\frac{1}{2}}$ there exists an unique solution $u = L_q^{-1} f$ to the problem* (1)*, obeying the following bound:*

$$\|u\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|L_q^{-1}\| \, \|f\|_{\dot{H}^{-\frac{1}{2}}}.$$

Moreover, the operator norm only depends on the L^2-norm of q:

$$\|L_q^{-1}\| \lesssim C(\|q\|_2).$$

Solutions of equation (1) are related to the Scattering Transform used in [1] to study the defocusing Davey-Stewartson II equation. New bounds on $\bar{\partial}^{-1}$ and on pointwise multiplication are needed to apply Inverse-Scattering methods in this settings.

1.2 New bounds on fractional integrals

In the following, Mf is the Hardy-Littlewood maximal function.

Theorem 2. *Let $\alpha \in (0,d)$, and $p \in (1,2]$. For any $f \in L^p(\mathbb{R}^d)$ we have:*

a) $|(-\Delta)^{-\frac{\alpha}{2}} f(x)| \lesssim_{d,\alpha} \lambda^{d-\alpha} M\hat{f}(0) + \lambda^{-\alpha} Mf(x)$ *for any* $\lambda > 0$;

b) $|(-\Delta)^{-\frac{\alpha}{2}} f(x)| \lesssim_{d,\alpha} \Big(M\hat{f}(0)\Big)^{\frac{\alpha}{d}} \Big(Mf(x)\Big)^{1-\frac{\alpha}{d}}$.

In order to apply the result to the Scattering transform, we rewrite point *b)* using $e^{iy\xi} f(y)$ as function of y in place of f. Then

b) $|(-\Delta)^{-\frac{\alpha}{2}} \big(e^{iy\xi} f(y)\big)(x)| \lesssim_{d,\alpha} \Big(M\hat{f}(\xi)\Big)^{\frac{\alpha}{d}} \Big(Mf(x)\Big)^{1-\frac{\alpha}{d}}$.

We are mainly interested in the case $d = 2$, $\alpha = 1$.

Corollary 3. *For $q \in L^2(\mathbb{C})$ we have*

b') $|\bar{\partial}^{-1}(e^{-i(zk+\overline{zk})} q(z))(x)| \lesssim \Big(M\hat{q}(k)\Big)^{\frac{1}{2}} \Big(Mq(x)\Big)^{\frac{1}{2}}$

c) $\|\bar{\partial}^{-1}(e^{-i(zk+\overline{zk})} q(z))\|_{L^4} \lesssim \|q\|_{L^2}^{\frac{1}{2}} \Big(M\hat{q}(k)\Big)^{\frac{1}{2}}$.

We use Theorem 2 to show L^2-boundness for a class of pseudo-differential operators (PDOs) with non-smooth symbols.

Theorem 4. *Let $\alpha \in [0,d)$, and $a(x,\xi)$ be a symbol on $\mathbb{R}^d \times \mathbb{R}^d$ such that*

i) $a \in L^{\frac{2d}{d-\alpha}}(\mathbb{R}^d \times \mathbb{R}^d)$, *and*

ii) $\|(-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\xi)\|_{L_\xi^{\frac{2d}{d+\alpha}}} \in L_x^{\frac{2d}{d-\alpha}}$

then the pseudo-differential operator

$$a(x,D)f(x) := \int_{\mathbb{R}^d} e^{ix\xi} a(x,\xi) \hat{f}(\xi) \frac{d\xi}{(2\pi)^d}$$

is bounded on L^2. We have the following bounds:

$$\|a(x,D)f\|_{L^2} \lesssim_{\alpha,d} \|(-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\xi)\|_{L_x^{\frac{2d}{d-\alpha}} L_\xi^{\frac{2d}{d+\alpha}}} \|f\|_{L^2}$$

$$|a(x,D)f(x)| \lesssim_{\alpha,d} (Mf(x))^{\frac{\alpha}{d}} \|(-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\cdot)\|_{L_\xi^{\frac{2d}{d+\alpha}}} \|f\|_{L^2}^{1-\frac{\alpha}{d}} \quad \text{for a.e. } x \in \mathbb{R}^d.$$

1.3 Estimates on pointwise multiplier

By the Sobolev embedding $\dot{H}^r(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d)$, with $p^* = \frac{2d}{d-2r}$. We embed the dual space $L^{(p^*)'}(\mathbb{R}^d) \hookrightarrow \dot{H}^{-r}(\mathbb{R}^d)$. To show the continuity of the map

$$\begin{aligned} \dot{H}^r(\mathbb{R}^d) &\to \dot{H}^{-r}(\mathbb{R}^d) \\ u &\mapsto qu \end{aligned}$$

it is enough to prove that it maps continuously L^{p^*} into its dual. This follows from the embeddings above and Hölder's inequality:

$$\|qu\|_{\dot{H}^{-r}} \lesssim \|qu\|_{(p^*)'} \le \|q\|_p \|u\|_{p^*} \lesssim \|q\|_p \|u\|_{\dot{H}^r}. \tag{2}$$

It gives $q \in L^p$, with $p = d/2r$. In our case ($\mathbb{C} \cong \mathbb{R}^2$) from (2) we have

$$\|qu\|_{\dot{H}^{-r}(\mathbb{R}^2)} \lesssim \|q\|_{L^{\frac{1}{r}}} \|u\|_{\dot{H}^r(\mathbb{R}^2)}.$$

We can improve the above estimate, trading regularity with integrability, by putting q in a negative homogeneous Besov space. The norm of $\dot{B}^{s,p}_q$ is

$$\|f\|_{\dot{B}^{s,p}_q} = \left\| 2^{sk} \|P_k f\|_{L^p} \right\|_{\ell^q}$$

where P_k is the Littlewood-Paley projector. We have the following theorem.

Theorem 5. *Let* $r \in [0,1)$ *and* $\max\left\{2, \frac{d}{r}\right\} \le p < \frac{d}{2r}$. *Then*

$$\|qu\|_{\dot{H}^{-r}(\mathbb{R}^d)} \lesssim \|q\|_{\dot{B}^{\frac{d}{p}-2r}_{p,\infty}} \|u\|_{\dot{H}^r(\mathbb{R}^d)}.$$

Note: The space $\dot{B}^{\frac{d}{p}-2r}_{p,q}$ has the same scaling of $L^{\frac{d}{2r}}$, but negative regularity.

Sketch of the proof. Write

$$qu = \sum_{(k_1,k_2,k_3) \in \mathcal{A}} P_{k_1}\left((P_{k_2} q)(P_{k_3} u) \right),$$

where P_k is the frequency projector to $A_k = \{2^{j-1} < |\xi| < 2^{j+1}\}$, and

$$\mathcal{A} = \{(k_1,k_2,k_3) \in \mathbb{Z}^3 \ : \ A_{k_1} \cap (A_{k_2} + A_{k_3}) \neq 0\}.$$

Estimate $\|qu\|_{\dot{H}^{-r}}$ using Bernstein inequalities and Littlewood-Paley trichotomy. □

1.4 A $\bar\partial$-problem

In order to prove Theorem 1, we have to show that, for $q \in L^2(\mathbb{C})$, L_q is invertible from $\dot H^{\frac12}(\mathbb{C})$ to $\dot H^{-\frac12}(\mathbb{C})$. We recall two known bounds on $\bar\partial^{-1}$:

Lemma 6. i) *Let $p \in (1,2)$, and $1/p^* = 1/p - 1/2$. For $h \in L^p$ we have*

$$\|\bar\partial^{-1} h\|_{L^{p^*}} \lesssim_p \|h\|_{L^p} \tag{3}$$

ii) *Let $1 < p_1 < 2 < p_2$ and $f \in L^{p_1} \cap L^{p_2}$, then*

$$\|\bar\partial^{-1} f\|_\infty \lesssim_{p_1,p_2} \|f\|_{L^{p_1}} + \|f\|_{L^{p_2}}.$$

The inverse operator L_q^{-1} is well defined from $L^{\frac43}$ to L^4.

Lemma 7. *Let $q \in L^2(\mathbb{C})$. Then $L_q u = f$ has an unique solution for $f \in L^{\frac43}$. In particular, the operator $L_q \colon L^4 \to L^{\frac43}$ is invertible.*

Idea. By the previous result, $\bar\partial^{-1} \colon L^{\frac43} \to L^4$ continuously. We write

$$L_q = \bar\partial(I + \bar\partial^{-1}(q\,\bar\cdot\,)) =: \bar\partial \circ \mathcal{B}.$$

Then it is enough to show the existence of an unique solution to $\mathcal{B}u = \bar\partial^{-1} f$ for $f \in L^{\frac43}$. In other words, if $\mathcal{B}\colon L^4 \to L^4$ is invertible, the unique solution to $L_q u = f$ is given by $u = \mathcal{B}^{-1}\bar\partial^{-1} f$.

Proof. Since the operator $\bar\partial^{-1}(q\,\bar\cdot\,)$ is compact from L^4 to itself, the operator $\mathcal{B} := (I+\bar\partial^{-1}(q\,\bar\cdot\,))$ is Fredholm, in particular $\mathcal{B}$ is injective iff is surjective.We prove that $\mathcal{B}$ is injective. Let $u \in L^4$ such that $\mathcal{B}u = 0$, i.e. $\bar\partial u = -q\bar u$. Write $q = q_n + q_s$, where q_s has small L^2-norm to be determined. We can choose $\nu \in L^\infty$ such that (!) holds[1] in the following

$$\bar\partial(u\nu) = (\bar\partial u)\nu + u\bar\partial\nu \overset{(!)}{=} (\bar\partial u + q_n \bar u)\nu \overset{(*)}{=} (-q_s\bar u)\nu$$

where $(*)$ holds since $\bar\partial u = -q\bar u$. Then, using bound (3) on $\bar\partial^{-1}$, we have

$$\|u\nu\|_{L^4} \le c\|\bar\partial(u\nu)\|_{L^{\frac43}} = c\|q_s\bar u\nu\|_{L^{\frac43}} \le c\|q_s\|_{L^2}\|u\nu\|_{L^4} \le \frac12\|u\nu\|_{L^4}$$

where in the last inequality we chose q_s with $\|q_s\|_{L^2} \le (2c)^{-1}$. This shows that, if $\mathcal{B}(u\nu) = 0$, $u\nu = 0$, so $\ker(\mathcal{B}) = \{0\}$. □

[1]Choose ν as solution of the equation

$$\bar\partial\nu = q_n\frac{\bar u}{u}\nu.$$

The same result holds when we consider L_q^{-1} on Sobolev spaces.

Lemma 8. *For $q \in L^2(\mathbb{C})$ the operator $L_q\colon \dot{H}^{\frac{1}{2}} \to \dot{H}^{-\frac{1}{2}}$ is invertible and*

$$\|L_q^{-1} f\|_{\dot{H}^{\frac{1}{2}}} \leq C(q)\|f\|_{\dot{H}^{-\frac{1}{2}}}.$$

We now study the dependence of L_q^{-1} and $C(q)$ on q.

Lemma 9. *The constant $C(q)$ has a local Lipschitz dependence on q. Given $q_0 \in L^2$, there exists $\epsilon > 0$ such that for every $q_1, q_2 \in B(q_0, \epsilon)$.*

$$\|L_{q_1}^{-1} - L_{q_2}^{-1}\| \lesssim C(q_0)^2\|q_1 - q_2\|_{L^2}$$
$$|C(q_1) - C(q_2)| \lesssim C(q_0)^2\|q_1 - q_2\|_{L^2}.$$

It remains to prove that the bound on $C(q)$ is uniform for q in a bounded set in L^2. Denote with

$$C(R) := \sup\{C(q) \,:\, \|q\|_2 \leq R\}, \qquad C\colon \mathbb{R}_+ \to [0,\infty].$$

The previous lemma, taking $q_0 = 0$, showed that $C(R)$ is finite for small R. We shall prove that it is finite for all $R > 0$. Argue by contradiction: let

$$R_0 := \inf\{R \in \mathbb{R}_+ \,:\, C(R) = +\infty\}.$$

Then $\lim_{R\to R_0} C(R) = +\infty$, and there exists a sequence $\{q_n\}_{n\in\mathbb{N}} \subset B_{R_0}$ such that $\|q_n\|_2 \to R_0$, with $\|L_{q_n}^{-1}\| \overset{n\to\infty}{\longrightarrow} +\infty$. If we were able to extract a convergent subsequence $\{q_{n_k}\}$ we would have

$$q_{n_k} \overset{L^2}{\to} q, \;\text{ and }\; \|L_{q_k}^{-1}\| \overset{k\to\infty}{\longrightarrow} \|L_q^{-1}\| < +\infty$$

leading to a contradiction, since R_0 was minimal. Unfortunately, we cannot hope to extract a subsequence converging in L^2.

Symmetries: obstruction to compactness Translations and dilations are symmetries of the problem that preserve the L^2-norm. Indicate with $S(\lambda, y)q(x) = \lambda q(\lambda(x - y))$. One has

$$C(q) = C(S(\lambda, y)q).$$

To prove finiteness of $C(R)$, it would suffices to show *compactness up to symmetries* of $\{q_n\}$ in a *weaker* topology.

Definition 10. *The sequence* $\{q_n\}$ *is compact up to symmetries if there exists a sequence* $\{(\lambda_n, y_n)\}_{n\in\mathbb{N}}$ *such that* $\{S(\lambda_n, y_n)q_n\}$ *is compact.*

Using Theorem 5, we can extend the result of Lemma 9 to a larger space: the Besov space $\dot{B}_{\infty}^{-\frac{1}{3},3}$.

Lemma 11. *Given* $q_0 \in L^2$*, there exists* $\epsilon = \epsilon(C(q_0)) > 0$ *such that for* $q_1, q_2 \in \{q : \|q - q_0\|_{B_{\infty}^{-\frac{1}{3},3}} < \epsilon\}$ *we have*

$$\|L_{q_1}^{-1} - L_{q_2}^{-1}\| \lesssim C(q_0)^2 \|q_1 - q_2\|_{B_{\infty}^{-\frac{1}{3},3}}$$
$$|C(q_1) - C(q_2)| \lesssim C(q_0)^2 \|q_1 - q_2\|_{B_{\infty}^{-\frac{1}{3},3}}.$$

Using profile decomposition we can split $\{q_n\}$ in different pieces driven by different symmetries and conclude the proof.

References

[1] Adrian Nachman, Idan Regev, and Daniel Tataru, *A Nonlinear Plancherel Theorem with Applications to Global Well-Posedness for the Defocusing Davewy-Stewartson Equation and to the Inverse Boundary Value Problem of Calderon.*, arXiv preprint arXiv:1708.04759 (2017).

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Extras

Why is ν bounded? Consider the case $\nu \neq 1$, otherwise both ν and $1/\nu$ are clearly in L^∞. Then $\nu := e^{\bar\partial^{-1}(q_n \frac{\bar u}{u})}$, with $q_n \in L^{p_1} \cap L^{p_2}$. Then

$$\|e^{\bar\partial^{-1}(q_n \frac{\bar u}{u})}\|_\infty \cong \left\| \sum_{k\geq 0} \frac{(\bar\partial^{-1}(q_n))^k}{k!} \right\|_\infty \leq \sum_{k\geq 0} \frac{\|\bar\partial^{-1} q_n\|_\infty^k}{k!} \lesssim_{p_1,p_2} \sum_{k\geq 0} \frac{(\|q_n\|_{p_1} + \|q_n\|_{p_2})^k}{k!}$$

that is finite and it equals $e^{\|q_n\|_{p_1} + \|q_n\|_{p_2}}$.

The embedding $\mathbf{L^2 \hookrightarrow \dot{B}_\infty^{-\frac{1}{3},3}}$ If

$$\dot{B}_1^{\frac{1}{3},\frac{3}{2}}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2) \tag{4}$$

then by duality

$$L^2 \hookrightarrow \dot{B}_\infty^{-\frac{1}{3},3}.$$

Checking (4) is straightforward, using the following embedding:

Lemma 12. *If $q \leq r$, then $\dot{B}_q^{s,p} \hookrightarrow \dot{B}_r^{s,p}$.*

Proof. This follows from the inclusion $\ell^q \hookrightarrow \ell^r$ for $q \leq r$. □

Note that $L^2 \cong \dot{B}_2^{0,2}$. By Lemma 12, one has

$$\begin{aligned} \|f\|_{L^2} = \left(\sum_{k\in\mathbb{Z}} \|P_k f\|_{L^2}^2 \right)^{\frac{1}{2}} &\leq \sum_k \|P_k f\|_{L^2} \\ &\leq \sum_k 2^{2k\left(\frac{2}{3}-\frac{1}{2}\right)} \|P_k f\|_{L^{\frac{3}{2}}} =: \|f\|_{\dot{B}_1^{\frac{1}{3},\frac{3}{2}}} \end{aligned}$$

where in the last inequality we used Bernstein's inequality:

$$\|P_k^2 f\|_{L^q(\mathbb{R}^d)} \leq 2^{kd(\frac{1}{p}-\frac{1}{q})} \|P_k f\|_{L^p(\mathbb{R}^d)}.$$

Proof of Lemma 9. Write the equation $L_q u = f$ as

$$\bar\partial u + q_0 \bar u + (q - q_0)\bar u = f, \quad \text{so that } u = L_{q_0}^{-1}\left((q_0 - q)\bar u + f\right).$$

Estimate

$$\begin{aligned}\|u\|_{\dot H^{\frac12}} &\le \|L_{q_0}^{-1}((q_0-q)\bar u)\|_{\dot H^{\frac12}} + \|L_{q_0}^{-1} f\|_{\dot H^{\frac12}}\\ &\lesssim C(q_0)\|(q_0-q)\bar u\|_{\dot H^{-\frac12}} + C(q_0)\|f\|_{\dot H^{-\frac12}}\\ &\lesssim C(q_0)\|q_0-q\|_{L^2}\|u\|_{\dot H^{\frac12}} + C(q_0)\|f\|_{\dot H^{-\frac12}}\end{aligned}$$

If $\|q_0-q\|_{L^2} \ll C(q_0)^{-1}$, then

$$\|u\|_{\dot H^{\frac12}} \lesssim 2C(q_0)\|f\|_{\dot H^{-\frac12}}.$$

Under the above condition, we also have

$$\begin{aligned}\|L_q^{-1} f - L_{q_0}^{-1} f\|_{\dot H^{\frac12}} &= \|L_{q_0}^{-1}((q_0-q)\bar u)\|_{\dot H^{\frac12}}\\ \text{(as above)}\quad &\lesssim C(q_0)\|q_0-q\|_{L^2}\|u\|_{\dot H^{\frac12}}\\ &\lesssim C(q_0)^2\|q_0-q\|_{L^2}\|f\|_{\dot H^{-\frac12}}.\end{aligned}$$

This implies that, for $\epsilon>0$ small enough, for q_1, q_2 in a ball $B(q_0,\epsilon)$

$$\|L_{q_1}^{-1} - L_{q_2}^{-1}\|_{\dot H^{-\frac12}\to \dot H^{\frac12}} \le C(q_0)^2\|q_2-q_1\|_{L^2}.$$

□

Proof of Theorem 2. Use Littlewood-Paley decomposition and write

$$(-\Delta)^{-\frac{\alpha}{2}} f(x) = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^d} \frac{e^{ix\xi}}{|\xi|^\alpha}\psi_j(\xi)\hat f(\xi)\frac{d\xi}{(2\pi)^d}$$

where

$$\operatorname{supp}(\psi_j) \subseteq A_j := \{\xi \,:\, 2^{j-1} < |\xi| < 2^{j+1}\}. \tag{5}$$

Fix $j_0\in\mathbb{Z}$ and split the sum $\sum = \sum_{j=-\infty}^{j_0} + \sum_{j=j_0+1}^{+\infty} = I + II$. For the first term

$$\begin{aligned} I &\lesssim \sum_{j=-\infty}^{j_0} 2^{(1-j)\alpha}\int_{\mathbb{R}^d} e^{ix\xi}\psi_j(\xi)\hat f(\xi)d\xi\\ &\lesssim_{d,\alpha} \sum_{j=-\infty}^{j_0} 2^{j(d-\alpha)}\int_{\{|\xi|<2^{j+1}\}} |\hat f(\xi)|d\xi \lesssim_{d,\alpha} 2^{j_0(d-\alpha)} M\hat f(0).\end{aligned}$$

For the second term, denote with $f^\vee$ the inverse Fourier transform, then

$$II \sim \sum_{j=j_0+1}^{+\infty} \left(\left(\frac{\psi_j(\xi)}{|\xi|^\alpha} \right)^\vee * f \right)(x) = (K_j * f)(x)$$

Split again the sum in j and bound by $2^{-j\alpha} Mf(x)$. Then the sum is bounded

$$II \lesssim \sum_{j=j_0+1}^{+\infty} 2^{-j\alpha} Mf(x) \lesssim 2^{-j_0\alpha} Mf(x).$$

The first statement follows by taking $\lambda = 2^{-j_0}$, while the second by optimising over λ. $\square$