Nodal Domains and Landscape Functions

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1 The Stoïlow factorisation theorem

After Astala, Iwaniec, and Martin [AIM09, §5]

A summary written by Gianmarco Brocchi

Abstract

We give a proof of the Stoïlow factorisation theorem for quasiconformal mappings on the plane. This result is then used to obtain a classification of quasiconformal mappings, and so uniqueness of (normalised) solutions to the Beltrami equation.

1.1 Introduction

The Stoïlow factorisation theorem says that two different solutions to the Beltrami equation are related by a holomorphic function.

1.1.1 Quasiconformal mappings

Let μ be a measurable function on $\Omega \subset \mathbb{C}$ with small L^∞-norm $\|\mu\|_\infty = \varepsilon < 1$. The weak solutions (in $H^1_{\text{loc}}(\Omega)$) to the Beltrami equation

$$\frac{\partial}{\partial \bar{z}} f(z) = \mu(z) \frac{\partial}{\partial z} f(z) \qquad \text{a. e. } z \in \Omega \subset \mathbb{C} \tag{B}$$

that are *homeomorphisms* are called quasiconformal mappings. Geometrically, these are maps of "bounded distorsion" on the plane.

The following result relates a homeomorphic solution to (B) and any other solution.

Theorem 1 (Stoïlow factorization)**.** *Let $\Omega \subset \mathbb{C}$, and let $f, g \in W^{1,2}_{loc}(\Omega)$ be two solutions to the same Beltrami equation* (B) *with f a quasiconformal map. Then there exists a holomorphic map Φ on $f(\Omega)$ such that*

$$g(z) = \Phi(f(z)) \qquad \textit{for all } z \in \Omega.$$

Moreover, for any holomorphic function Φ on $f(\Omega)$, the map $\Phi \circ f$ is a solution of (B)*.*

The Stoïlow factorisation can be used to parameterise solutions to (B) on the whole plane by their value at different points.

Corollary 2. *Let $f, g \in W^{1,2}_{loc}(\Omega)$ be two homeomorphic solutions to* (B) *on* $\mathbb{C}$. *If f and g fix the points 0 and 1, then $f = g$.*

Proof of the Corollary. By Stoïlow factorisation, there exists an entire function Φ such that $g = \Phi \circ f$. Since both f and g are homeomorphism, Φ has to be injective, so in particular Φ is conformal. Entire conformal maps are similarities (they preserve the ratio of distances), and a similarity which fixes 0 and 1 (and $\Phi(\infty) = \infty$) is the identity. □

We say that a quasiconformal homeomorphism $f : \mathbb{C} \to \mathbb{C}$ is *normalised* if it fixes the origin and the point 1. Then the corollary above implies that normalised solutions to the Beltrami equation are unique.

Before going into the proof of the Theorem 1, we recall a few useful facts needed to find solutions to the Beltrami equation.

1.2 Solving the Beltrami equation

To find solutions to (B), we start by assuming that the Beltrami coefficient μ is smooth and compactly supported. Then, consider the inhomogeneous equation:

$$\frac{\partial}{\partial \bar{z}}\sigma = \mu(z)\frac{\partial}{\partial z}\sigma + \varphi \tag{1}$$

where $\varphi \in L^p(\mathbb{C})$ and compactly supported.

1.2.1 Cauchy and Beurling transform

A couple of operators are relevant to us: the Cauchy transform $\mathcal{C} := (\partial/\partial\bar{z})^{-1}$, mapping $\mathcal{C} : L^p(\mathbb{C}) \to W^{1,p}(\mathbb{C})$ for $p > 2$, which is the singular integral operator given by

$$\mathcal{C}f(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{z - \zeta} d\zeta,$$

and the Beurling transform S, which is $\partial/\partial z \circ \mathcal{C}$, so it is given by

$$Su(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{u(\zeta)}{(z - \zeta)^2} d\zeta.$$

Remark 3. *The operator S is bounded on $L^p(\mathbb{C})$ and it exchanges the weak derivative $\partial/\partial\bar{z}$ with $\partial/\partial z$, in particular $S(\sigma_{\bar{z}}) = \sigma_z$ for $\sigma \in W^{1,p}(\mathbb{C})$.*

Let $u \coloneqq (\partial/\partial\bar{z})\sigma$. By using the Beurling transform S, the inhomogeneous equation (1) is equivalent to

$$u = \mu(z)Su + \varphi.$$

The solution can be written as $u = (I - \mu S)^{-1}\varphi$, where we used the inverse of the Beltrami operator $(I - \mu(z)S)$. Indeed, the Neumann series of $(I - \mu(z)S)^{-1}$ converges if $\|\mu\|_\infty \|S\|_{L^p} < 1$. Since $\|\mu\|_\infty = \varepsilon < 1$, the Beltrami operator $I - \mu(z)S$ is invertible in a range of p where $\|S\|_p < 1/\varepsilon$.

Despite the fact that the exact value of $\|S\|_{L^p}$ is still unknown, it is a deep result that the maximal range of invertibility of $I - \mu(z)S$ is

$$I_\varepsilon \coloneqq \left(1 + \varepsilon, 1 + \frac{1}{\varepsilon}\right).$$

Back to our inhomogeneous problem, a solution to (1) is given by the Cauchy transform of u:

$$\sigma = \left(\frac{\partial}{\partial\bar{z}}\right)^{-1}(I - \mu(z)S)^{-1}\varphi.$$

Given a solution to the inhomogeneous problem (1), a solution to the original Beltrami equation (B) is $f = z + \sigma$, by taking $\varphi = \mu$ in (1).

Now one can show that solution are smooth by a bootstrapping argument: first, taking $\mu = \varphi \in W^{1,p}(\mathbb{C})$ implies $\sigma \in W^{2,p}(\mathbb{C})$. Then the smoothness of μ implies $\sigma \in C^\infty$, and so the smoothness of f.

For the general case (μ only measurable with small L^∞ norm) one can solve (B) with the smooth approximation $\mu_\delta \coloneqq \mu * \phi_\delta \to \mu$ as $\delta \to 0$, and ϕ smooth. Then exploit the compactness properties of the class of quasiconformal mappings and the boundedness of the Cauchy transform to prove the uniform convergence of the approximate solution f_δ. Details are in [AIM09, §5.3].

We now move to the proof of the factorisation theorem.

1.3 Proof of the Stoïlow factorisation

The idea of the proof is to show that the map

$$\Phi \coloneqq g \circ f^{-1}$$

is continuous and then that is holomorphic. Note that g is not a priori continuous, but we will see that any $W^{1,2}_{\text{loc}}$-solution to the Beltrami equation (B) is indeed continuous. We will prove this result later. For the time being, we will assume the g is continuous and it has weak derivative in L^2.

Then it follows that Φ is continuous, because f is a homeomorphism. We want to show that $\partial/\partial\bar{w}\Phi \equiv 0$. By the chain rule:

$$\frac{\partial}{\partial \bar{w}}\Phi(w) = (g_z \circ f^{-1})(w)\frac{\partial}{\partial \bar{w}}(f^{-1})(w) + (g_{\bar{z}} \circ f^{-1})(w)\overline{\frac{\partial}{\partial w}(f^{-1})(w)}.$$

Since it can be seen that inverse function f^{-1} satisfies the equation

$$\frac{\partial}{\partial \bar{w}}(f^{-1}) = -\mu(f^{-1}(w))\overline{\frac{\partial}{\partial w}f^{-1}}$$

by rewriting the chain rule above, we have that

$$\frac{\partial}{\partial \bar{w}}\Phi = \overline{\frac{\partial}{\partial w}f^{-1}}\big[-g_z(f^{-1})\mu(f^{-1}) + g_{\bar{z}}(f^{-1})\big] = 0$$

because g satisfies the Beltrami equation.

To conclude, we invoke the Weyl's lemma, which states that weak solution to $\partial/\partial\bar{w}$ in $L^1_{\text{loc}}(\mathbb{C})$ are analytic. This shows that Φ is holomorphic and concludes the proof. □

It is left to show that $W^{1,2}_{\text{loc}}$-solutions to the Beltrami equation are continuous. This result exploits the L^p mapping property of the Beltrami operator presented above. In particular, when $\|\mu\|_\infty < \varepsilon < 1$, the inverse of the Beltrami operator is continuous on L^p for p in the range $I_\varepsilon := (1+\varepsilon, 1+1/\varepsilon)$.

1.3.1 Continuity of $W^{1,2}_{\text{loc}}$ solutions

Theorem 4. *Let $\Omega \subset \mathbb{C}$. Let $f \in W^{1,2}_{loc}(\Omega)$ be a solution to* (B) *with $\|\mu\|_\infty = \varepsilon < 1$. Then $f \in W^{1,p}_{loc}(\Omega)$ for all $p \in I_\varepsilon$.*

In particular, $f \in W^{1,2+s}_{\text{loc}}(\Omega)$ for some $s > 0$, so by the Sobolev embedding f is continuous.

Sketch of the proof of Theorem 4. Consider the function ψf, for $\psi \in C^\infty_c(\Omega)$. Since f is a solution to the Beltrami equation, by the chain rule we have

$$(\psi f)_{\bar{z}} - \mu(\psi f)_z = f \cdot (\psi_{\bar{z}} - \mu\psi_z) =: \varphi.$$

By solving the inhomogeneous Beltrami equation for $F = \psi f$

$$F_{\bar{z}} = \mu F_z + \varphi$$

we find expressions for the weak derivative of F, that are

$$\begin{aligned} F_{\bar{z}} &= (I - \mu S)^{-1}\varphi \\ F_z &= SF_{\bar{z}} = S \circ (I - \mu S)^{-1}\varphi. \end{aligned}$$

The Sobolev membership of f then follows from the L^p boundedness of the partial derivative $F_{\bar{z}}, F_z$, which is a consequence of the L^p mapping property of the Beurling transform S and $(I - \mu S)^{-1}$ for p in the range I_ε. □

References

[AIM09] Kari Astala, Tadeusz Iwaniec, and Gaven Martin. *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane.* (PMS-48). Princeton University Press, 2009.

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