Brascamp–Lieb inequalities

September 2021

Organizers:

Christoph Thiele, Universität Bonn

Pavel Zorin-Kranich, Universität Bonn

Contents

1 The Kempf–Ness theorem

After Böhm and Lafuente [BL]

A summary written by Gianmarco Brocchi

Abstract

The Brascamp–Lieb constant is related to the length of minimal vectors in the sense of the Kempf–Ness theorem. We present the real version of the theorem and main ideas of the proof.

1.1 Introduction

Given a collection of surjective maps $\pi_j \colon \mathbb{R}^d \to \mathbb{R}^{d_j}$ and numbers $s_j > 0$ for $j \in \{1, \dots, m\}$, with m, d and $d_j \in \mathbb{N}$, with $d_j < d$, we consider the Brascamp–Lieb inequality:

$$\int_{\mathbb{R}^d} \prod_j |f_j(\pi_j x)|^{s_j} dx \leq \mathsf{BL}(\{\pi_j, s_j\}) \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j(y) dy \right)^{s_j} . \tag{1}$$

The inequality (1) is maximised by Gaussian. Let g_j be a Gaussian on $\mathbb{R}^{d_j}$. By plugging g_j into (1) we obtain

$$\mathsf{BL}(\{\pi_j, s_j\}) \geq \frac{\int_{\mathbb{R}^d} \prod_j |g_j(\pi_j x)|^{s_j} dx}{\prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} g_j(y) dy \right)^{s_j}} = \left(\frac{\det(\sum_j s_j \pi_j^* A_j^* A_j \pi_j)}{\prod_{j=1}^m \det(A_j^* A_j)^{s_j}} \right)^{-1/2} .$$

So the optimal constant $\mathsf{BL}(\{\pi_j, s_j\})$ is achieved by taking the supremum over all matrices $A_j \in \mathsf{GL}(d_j)$. In [Gr], the right hand side of the expression above is written by using the Hilbert–Schmidt norm, so that

$$\mathsf{BL}(\{\pi_j, s_j\})^{-1} = \inf_{\substack{A_j \in \mathsf{SL}(d_j) \\ A \in \mathsf{SL}(d)}} \prod_{j=1}^m \left(d_j^{-1/2} \| A_j \pi_j A^* \|_{\mathrm{HS}} \right)^{s_j d_j} . \tag{2}$$

Equation (2) gives a way to approximate Brascamp–Lieb constant by minimising a "distance function" under the action of the group $G \subset \mathsf{GL}(d_1) \otimes \cdots \otimes \mathsf{GL}(d_m) \otimes \mathsf{GL}(d)$ on the vector space $(V, \langle \cdot, \cdot \rangle)$ where the projections π_j live. The quantity $\mathsf{BL}(\{\pi_j, s_j\})^{-1}$ is the length of the minimal vector in a given orbit.

A classical theorem by George Kempf and Linda Ness relates closed orbits and minimal vectors.

1.2 Kempf–Ness theorem

Definition 1. *Let G be a group acting on a vector space V endowed with inner product $\langle \cdot, \cdot \rangle$, and let $d\colon V \to \mathbb{R}_+$ be a given function. For $v \in V$, a* minimal *vector $\bar{v}$ in the orbit $G \cdot v$ is a vector that minimises $d(\cdot)$.*

Let $\mathscr{M}$ be the set of minimal vectors in V.

Remark 2 (Closed orbits intersect $\mathscr{M}$)**.** *If the orbit $G \cdot v$ is closed (as a set), the intersection with closed balls $B_R(0) := \{w : d(w) \leq R\}$ for R large enough is not empty and is compact. In particular, $d(\cdot)$ has a minimum on $G \cdot v \cap B_R(0)$, so $G \cdot v$ contains a minimal vector.*

The converse, for real reductive Lie groups, is the (real version of the) Kempf–Ness theorem. We briefly introduce reductive Lie groups.

Let G be a Lie group and let $\mathfrak{g}$ be the Lie algebra of G. We will consider the symmetric part of the algebra given by the Cartan decomposition.

Remark 3 (Cartan decomposition)**.** *Let $G \subset \mathsf{GL}(d)$ be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Then $\mathfrak{g}$ can be decomposed in symmetric and anti-symmetric part: $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$, where $\mathfrak{s} = \mathfrak{g} \cap \mathsf{Sym}(V)$. If $[\cdot, \cdot]$ is the Poisson bracket: $[a, b] = ab - ba$, then $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{a}$ and $[\mathfrak{s}, \mathfrak{a}] \subset \mathfrak{s}$.*

Definition 4. *A Lie group G is called* reductive *if can be written as $G = K \cdot \exp(\mathfrak{s})$, where K is a maximal compact subgroup of G.*

We will be interested in subgroups of $\mathsf{GL}(d, \mathbb{R})$.

Theorem 5 (Real Kempf–Ness Theorem)**.** *Let $G \subset \mathsf{GL}(d, \mathbb{R})$ be a reductive Lie group with a maximal subgroup $K = G \cap O(\mathbb{R}^d)$. For $v \in V$ the orbit $G \cdot v$ contains a minimal vector if and only if is closed. Moreover $G \cdot v \cap \mathscr{M} = K \cdot v$.*

1.3 Proof of the theorem

The proof is based on two main facts:

1. If the orbit $G \cdot v$ is not closed, the elements in the closure can be reached with a one-parameter subgroup. This is proved by contradiction: assuming that all such orbits are separated leads to an absurd.

2. If the distance function $d(\cdot)$ is strictly convex, its critical points are minima and there are not such points on non-closed orbits.

We start by considering the simpler case of abelian groups.

1.4 Abelian case

Let T be a abelian, non compact, connected Lie group[1] and let $\mathfrak{t}$ be its Lie algebra. Let $K \subset T$ be a maximal, compact subgroup. In the complex case, one can think of K as the elements of T with modulus 1.

Representation

The elements in T can be written as $\exp(t\alpha) \in T$ for $\alpha \in \mathfrak{t}$ and $t \in \mathbb{R}$. In particular, there is $(v_1, \dots, v_N)$ basis of V which diagonalises the action of T. Then for $\lambda \in \mathfrak{t}$ we have

$$e^{\lambda} \cdot v = (e^{\langle \lambda, \alpha_1 \rangle} v_1, \dots, e^{\langle \lambda, \alpha_N \rangle} v_N), \qquad \text{for} \qquad \alpha_1, \dots, \alpha_N \in \mathfrak{t}.$$

For an abelian group T, the representation of its action is enough to show that, given any $\bar{v} \in \overline{T \cdot v} \setminus T \cdot v$, there is a one-parameter semigroup intersecting the orbit $T \cdot \bar{v}$.

Lemma 6 (Hilbert–Mumford for abelian groups)**.** *For any $\bar{v} \in \overline{T \cdot v} \setminus T \cdot v$ there exists $g \in T$ and $\alpha \in \mathfrak{t}$ such that $\lim_{s \to \infty} \exp(s\alpha) \cdot v = g \cdot \bar{v}$.*

Convexity of the distance function

For $\alpha \in \mathfrak{s}$ and $t \in \mathbb{R}$, consider the distance function

$$d(t) := d_{\alpha,v}(t) := \|\exp(t\alpha) \cdot v\|^2.$$

This is the square of the distance to the origin along the curve $\exp(t\alpha) \cdot v$ in G. The function $d(t)$ is convex, so its critical points are minima.

Lemma 7 (Convexity)**.** *For $A \in \mathfrak{s}$ and $v \in V$ the function $d_{\alpha,v}(t)$ is convex, in particular $d''(t) = 4\|A \cdot \exp(tA) \cdot v\|^2$.*

Let $\alpha \in \mathfrak{s}$ and assume that $\lim_{t \to \infty} \exp(t\alpha) \cdot v = \bar{v}$ exists. Then, by convexity of $d(t)$, we have that

$$\|e^{t\alpha} \cdot v\| > \|\bar{v}\|, \quad \forall t \in \mathbb{R}.$$

Thus the function $d(t)$ cannot achieve its minimum on a non-closed orbit.

[1] the notation comes from T being an algebraic torus in the complex case.

1.5 Real reductive groups

For general real reductive groups, one can write $G = KTK$, where K is compact and T is abelian. It is enough to show that the limit of the one-parameter subgroup exists.

Lemma 8 (Hilbert–Mumford for real reductive groups). *Let G be a real reductive group and let $v \in V$. If the orbit $G \cdot v$ is not closed then there exists $\alpha \in \mathfrak{s}$ such that $\lim_{s\to\infty} \exp(s\alpha) \cdot v$ exists.*

Idea of the proof. Let $\mathfrak{t} \subset \mathfrak{s}$ be the maximal abelian subalgebra. Since $G = KTK$, with $T = \exp(\mathfrak{t})$, it is enough to show that given $\bar{v} \in \overline{G \cdot v} \setminus G \cdot v$ there exists $g \in G$, $k \in K$ and $\alpha \in \mathfrak{t}$ such that $\lim_{s\to\infty} \exp(s\alpha) \cdot (k \cdot v) = g \cdot \bar{v}$.

Suppose by contradiction that the two orbits $\overline{G \cdot \bar{v}}$ and $\overline{T \cdot k \cdot v}$ are disjoint for all $k \in K$. Assume we can separate any of these closed orbits with a function f_k. Exploiting the compactness of K, we can extract finitely many functions for the job and construct a single function f which separates $\overline{TK \cdot v}$ and $\overline{G \cdot \bar{v}}$. Since $K \cdot \bar{v} \subset G \cdot \bar{v}$, we can then separate the orbits $\overline{TK \cdot v}$ and $K \cdot \bar{v}$. But this implies that $\bar{v} \notin K(\overline{TK \cdot v})$ and so $\bar{v} \notin \overline{G \cdot v}$, which is absurd. $\square$

We discuss separation of orbits in the next subsection.

1.5.1 Separation of closed orbits

Consider a subset of coordinate indices $I \subset \{1, \dots, N\}$ and let U_I be the subset of vectors whose non-zero coordinates belongs to I: $U_I = \{v \in V : v_i \neq 0$ if and only if $i \in I\}$.

Lemma 9. *The orbit $T \cdot v$ is closed if and only if there is a convex combination $\{\theta_i\}$ of $\{\alpha_i\}$ such that $\sum_i \theta_i \alpha_i = 0$.*

Given a closed orbit $\mathcal{O}_1$, consider the corresponding $\theta := \{\theta_i\}$ given by the above lemma. Define the function $f_\theta \colon V \to \mathbb{R}$ as

$$f_\theta(v) := \begin{cases} \prod_{i=1}^N v_i^{\theta_i} & \text{if } i \in I \\ 0 & \text{otherwise .} \end{cases}$$

The function f_θ is continuous. Moreover, by using the representation of the action of T, we see that f_θ is also T-invariant, indeed

$$f_\theta(\exp(\lambda) \cdot v) = \prod_{i=1}^N (e^{\langle \lambda, \alpha_i \rangle} v_i)^{\theta_i} = e^{\langle \lambda, \sum_i \alpha_i \theta_i \rangle} f_\theta(v) = f_\theta(v).$$

Intuitively, if two closed orbits are distinct, there must exist a zero combination of α_i for one orbit that is not zero for the other one. In other words, there exists θ such that the map f_θ separates the two orbits.

Lemma 10. *Let $\mathcal{O}_1, \mathcal{O}_2$ be two distinct, closed T-orbits. Then there exists $\theta = \{\theta_i\}$ such that $f_\theta(\mathcal{O}_1) \neq f_\theta(\mathcal{O}_2)$.*

References

[KN] Kempf, G. and Ness, L., *The length of vectors in representation spaces.* Algebraic geometry. Springer, Berlin, Heidelberg, 1979. 233–243.

[BL] Böhm, C. and Lafuente, R. A., *Real geometric invariant theory.* arXiv:1701.00643 (2017).

[Gr] Gressman, P. T., *L^p-improving estimates for Radon-like operators and the Kakeya-Brascamp-Lieb inequality.* Advances in Mathematics, **387**, (2021), 107831.

Gianmarco Brocchi, University of Birmingham
email: `gianmarcobrocchi@gmail.com`

A Complex Kempf–Ness

We briefly present the ideas from the original proof in as they provide intuition for the real case.

Let $\rho_v(g) := \|g \cdot v\|^2$ be the length of the vector v along the orbit $G \cdot v$. We want to show that if the orbit $G \cdot v$ is not closed, then $\inf_{g \in G} \rho_v(g)$ is not attained.

Remark 11. *Let K be a subgroup of G which preserves the norm $\|\cdot\|$. Then the function ρ_v is invariant under the (right) action of the stabiliser of v in G, and under the (left) action of K.*

Thus, for $k \in K$ and $s \in Stab(v)$ in G we have

$$\rho_v(k \cdot g \cdot s) = \rho_v(g).$$

As the function ρ_v is constant under the action of K and $Stab(v)$, we quotient by these subgroups. Then, by the universal construction, if π is the projection from $T \to T/K$, there exists a unique map $q_v \colon T/K \to \mathbb{R}$ such that $\rho_v = q_v \circ \pi$. We quotient again by the image of $Stab(v)$ in T/K and so we obtain another map $\tilde{q}_v \colon (T/K)/\pi(Stab(v)) \to \mathbb{R}$. Moreover, if ρ_v is a finite sum of $e^{A(\cdot)}$ where $A(\cdot)$ is an affine function, then q_v and $\tilde{q}_v$ have the same form. This allows to infer property of ρ_v from the map $\tilde{q}_v$ where we have quotient out subspaces where ρ_v is constant.

Now consider a non-closed orbit $T \cdot v$. Then there exists a one-parameter semigroup $e^{s\alpha}$, with $\alpha \in \mathfrak{t}$ the Lie algebra of T, which leads outside the orbit $T \cdot v$. Assume that the limit as s goes to infinity of $e^{s\alpha} \cdot v$ exists and equals $\bar{v} \in \overline{T \cdot v} \setminus T \cdot v$.

Then, by convexity, the function $\tilde{q}_v$ (and so q_v and ρ_v) cannot attain its minimum on the orbit.