Sphere Packings and Optimal Configurations

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1 The Bourgain–Milman theorem

A summary written by Constantin Bilz and Gianmarco Brocchi after Bourgain–Milman [2] and Nazarov [5]

Abstract

We present the Bourgain–Milman theorem on Mahler's conjecture. We explain both the original proof [2] based on the geometry of normed spaces and Nazarov's proof [5] based on Hörmander's theorem.

1.1 Introduction

Let $K \subset \mathbb{R}^n$ be a convex centrally symmetric bounded open and absorbing set and let $K^\circ = \{x \in \mathbb{R}^n : |\langle x, y\rangle| \leq 1 \text{ for all } y \in K\}$ be the *polar set* of K. Let vol denote n-dimensional volume and let B_n be the n-dimensional euclidean ball.

Consider the affine invariant quantity $\operatorname{vol} K \cdot \operatorname{vol} K^\circ$. It holds that

$$\frac{4^n}{(n!)^2} \leq \operatorname{vol} K \cdot \operatorname{vol} K^\circ \leq (\operatorname{vol} B_n)^2.$$

The upper bound is sharp and was obtained by Santaló [7], improving on the upper bound 4^n established earlier by Mahler [4]. The lower bound was also proved by Mahler and he conjectured that it can be improved to

$$\operatorname{vol} C_n \cdot \operatorname{vol} C_n^\circ = \frac{4^n}{n!} \leq \operatorname{vol} K \cdot \operatorname{vol} K^\circ \tag{1}$$

so that the symmetric hypercube C_n would be minimising. He proved this for $n = 2$. Partial progress towards (1) in higher dimensions has been made by several authors, see e.g. [1] and the citations in [2]. We will present two proofs of the following

Theorem 1 (Bourgain–Milman)**.** *There exists a constant $c > 0$ independent of the dimension d such that*

$$\operatorname{vol} K \cdot \operatorname{vol} K^\circ \geq c^n \operatorname{vol} C_n \cdot \operatorname{vol} C_n^\circ. \tag{2}$$

We remark that the largest known constant for which Theorem 1 holds is $c = \frac{\pi}{4}$ and this is due to Kuperberg [3].

1.2 The proof of Bourgain–Milman

Let μ_{n-1} be the normalized surface measure on the euclidean unit sphere S^{n-1}. We denote the norm on $\mathbb{R}^n$ with unit ball K by $\|\cdot\|_K$. We write $E(K)$ for the normed space $(\mathbb{R}^n, \|\cdot\|_K)$ and we write

$$M_K = \int_{S^{n-1}} \|x\|_K \, d\mu_{n-1}(x), \qquad d_K = \frac{\sup_{x\in S^{n-1}} \|x\|_K}{\inf_{x\in S^{n-1}} \|x\|_K}.$$

The (multiplicative) *Banach–Mazur distance* between $(\mathbb{R}^n, |\cdot|_2)$ and $E(K)$ is

$$d_{E(K)} = \inf\{d_{u(K)} \mid u : \mathbb{R}^n \to \mathbb{R}^n \text{ linear isomorphism}\}.$$

The proof is based on an analysis of the linear structure of the convex body K starting with the following result.

Proposition 2. *Let $\lambda \in (0,1)$. There exists a subspace F of $E(K)$ such that*

$$\dim F \geq \lambda n \quad \textit{and} \quad \|x\|_K \geq c(1-\lambda) M_{K^\circ}^{-1} |x| \quad \textit{for any } x \in F.$$

Proof sketch. We apply the isoperimetric inequality on S^{n-1} to the geodesic $\pi/4$-neighbourhood $A_{\pi/4}$ of the set $A = \{\|x\|_{K^\circ} \leq 2M_{K^\circ}\}$. For any $k < n$ we hence find a k-dimensional subspace F that has a large intersection with $A_{\pi/4}$, namely

$$\mu_{k-1}(A_{\pi/4} \cap F) \geq 1 - \frac{\mathrm{vol}_{n-2} S^{n-2}}{\mathrm{vol}_{n-1} S^{n-1}} \int_0^{\pi/4} \sin^{n-2} t \, dt.$$

If $\tau \sim 1 - k/n$ and $x \in F \cap S^{n-1}$, then this implies

$$\mu_{k-1}(A_{\pi/4} \cap F) > 1 - \mu_{k-1}(B_{\pi/4-\tau}(x))$$

where $B_\epsilon(x) \subset F \cap S^{n-1}$ is the ball with respect to geodesic distance. Then we have $F \cap S^{n-1} \subset A_{\pi/2-\tau}$. This implies the proposition. □

We will combine this with an upper bound on M_{K°. Such a bound is provided by the following result which is well-known in the geometry of Banach spaces.

Proposition 3. *There is a linear isomorphism $u : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$M_{u(K)} \cdot M_{u(K)^\circ} \leq C(1 + \log d_{E(K)})^2.$$

We can now prove the following "subspace of quotient" result.

Lemma 4. *Let $\lambda \in (0,1)$. Then there exists a subspace F of $\mathbb{R}^n$ and a quotient space G of F such that*

$$\dim G \geq \lambda n \quad \textit{and} \quad d_G \leq C(1-\lambda)^{-2}(1+\log d_{E(K)})^2.$$

Proof sketch. We apply Proposition 2 twice. First, we find a subspace F of $E(K)$ with $\dim F \geq \sqrt{\lambda} n$ and by duality

$$\|x\|_{K^\circ} \leq C(1-\sqrt{\lambda})^{-1} M_{K^\circ}|x| \quad \text{for any } x \in F^*.$$

Here F^* denotes the dual space of F. Secondly, we find a subspace G of F^* such that $\dim G \geq \lambda n$ and

$$\|x\|_{K^\circ} \geq c(1-\sqrt{\lambda}) M_K^{-1}|x| \quad \text{for any } x \in G.$$

Now we replace K by the $u(K)$ from Proposition 3 and use the definition of d_G to complete the proof. □

Sketch of proof of Theorem 1. Fix an integer N. For $n \leq N$ let $\mathcal{C}_n(t)$ be the class of convex bodies K in $\mathbb{R}^n$ for which $d_{E(K)} \leq t$. We write

$$\sigma(t) = \inf_{\substack{n \leq N \\ K \in \mathcal{C}_n(t)}} \left(\frac{\mathrm{vol}_n K \cdot \mathrm{vol}_n K^\circ}{(\mathrm{vol}_n B_n)^2} \right)^{1/n}.$$

Using Lemma 4 we will show in the talk that

$$\sigma(t) \geq c^{\frac{1}{\log t}} \sigma(C(\log t)^6)$$

with constants independent of N. This inequality implies a uniform lower bound for $\sigma(t)$ which proves the theorem. □

1.3 An alternative proof via Hörmander's theorem

We can prove (2) constructing an analytic function on $\mathbb{C}^n$ with good decay property. By the Paley–Wiener theorem, given any $g \in L^2(K^\circ)$ its Fourier transform $f(w) = \int_{K^\circ} g(v) e^{-i\langle w,v\rangle} \mathrm{d}v$ extends to an entire function on

$\mathbb{C}^n$. Applying Cauchy–Schwarz $|f(0)|^2 \leq \|g\|^2_{L^2(K^\circ)} \operatorname{vol} K^\circ$, and Plancherel $\|f\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^n \|g\|^2_{L^2(K^\circ)}$ we have the lower bound

$$\operatorname{vol} K^\circ \geq (2\pi)^n \frac{|f(0)|^2}{\|f\|^2_{L^2(\mathbb{R}^n)}}.$$

We want an entire function which $L^2(\mathbb{R}^n)$-norm is not too large compared with its value at the origin. We look for such a function in a Bergman space with Hörmander type weight, i.e. $L^2(\mathbb{C}^n, e^{-\varphi})$ where φ is plurisubharmonic.

Let T_K be the (horizontal) tube domain $\{x + iy : x \in \mathbb{R}^n, y \in K\}$ and consider the Bergman space $A^2(T_K) = \{\text{analytic functions on } T_K\} \cap L^2(T_K)$.

This is a Hilbert space with reproducing kernel

$$\mathcal{K}(z, w) = \int_{\mathbb{R}^n} \frac{e^{i\langle z-\bar{w}, v\rangle}}{\int_K e^{-2\langle x,v\rangle} \mathrm{d}x} \frac{\mathrm{d}v}{(2\pi)^n}.$$

An application of Cauchy–Schwarz gives

$$|f(0)|^2 = \left| \int_{T_K} \mathcal{K}(0, w) f(w) \mathrm{d}w \right|^2 \leq \int |\mathcal{K}(0, w)|^2 \int |f(w)|^2 = \mathcal{K}(0, 0) \|f\|^2_{A^2(T_K)}$$

from which we have the lower bound for $\mathcal{K}(0, 0)$

$$\frac{|f(0)|^2}{\|f\|^2_{A^2(T_K)}} \leq \mathcal{K}(0, 0) = \int_{\mathbb{R}^n} \frac{1}{\int_K e^{-2\langle x,v\rangle} \mathrm{d}x} \frac{\mathrm{d}v}{(2\pi)^n} \leq \frac{n!}{\pi^n} \frac{\operatorname{vol} K^\circ}{\operatorname{vol} K}$$

and the upper one by using the convexity of $x \mapsto e^{-\langle x,v\rangle}$ and optimising in v.

Up to affine linear transformations, we can assume that K contains the ball $B(0, r)$. By the John's ellipsoid theorem, $K \subset B(0, R)$ with $R/r \leq \sqrt{n}$. For any $t \in K^\circ$, the Hermitian product $z \mapsto \langle z, t\rangle$ maps T_K in the strip $S = \{\zeta \in \mathbb{C} : |\Im(\zeta)| < 1\}$, while the conformal map

$$\phi(\zeta) = \frac{4}{\pi} \frac{e^{\frac{\pi}{2}\zeta} - 1}{e^{\frac{\pi}{2}\zeta} + 1}$$

maps the strip S to the disk $D(0, \frac{4}{\pi})$. Consider the set

$$K_{\mathbb{C}} := \{z \in \mathbb{C}^n : |\langle z, t\rangle| \leq 1, \forall t \in K^\circ\} \subset T_K.$$

Note that $K_{\mathbb{C}}$ contains $\frac{1}{\sqrt{2}}(K + iK)$. It is enough to construct an analytic function inside $K_{\mathbb{C}}$. For this purpose we will use the Hörmander's theorem.

Definition 5. *A function* $\varphi\colon \Omega \subset \mathbb{C}^n \to \mathbb{R}$ *is* strictly plurisubharmonic *if there exists* $\tau > 0$ *such that*

$$\langle H(z)w, w\rangle \geq \tau |w|^2, \quad \forall w \in \mathbb{C}^n, \forall z \in \Omega$$

where H *is the Hermitian matrix* $H = \left(\dfrac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}\right)_{i,j=1}^n$.

Theorem 6 (Hörmander)**.** *Let* $\Omega \subset \mathbb{C}^n$ *be an open, pseudoconvex domain, and let* $\varphi\colon \Omega \to \mathbb{R}$ *be strictly plurisubharmonic for a* $\tau > 0$*. For any* $(0,1)$*-form* ω *on* Ω *with* $\bar{\partial}\omega = 0$*, there exists a solution* h *of* $\bar{\partial}h = \omega$ *in* Ω *satisfying*

$$\int_\Omega |h|^2 e^{-\varphi} \mathrm{d}z \leq \tau^{-1} \int_\Omega |\omega|^2 e^{-\varphi} \mathrm{d}z.$$

We take the plurisubharmonic function φ on a shrunk version of $K_{\mathbb{C}}$:

$$\varphi(z) = \frac{|\Im(z)|^2}{R^2} + \log \sup_{t \in K^\circ} |\phi(\langle z, t\rangle)|^{2n}.$$

The first term enforces the strict plurisubharmonicity on any ball of radius $\delta < R$ with $\tau = \delta^2/R^2$. The second term ensures that the function h promised by the theorem will vanish at 0, as soon as $\int |\omega|^2 e^{-\varphi}$ is finite. Indeed, since $\phi(0) = 0$ and $\phi'(0) = 1$, using Taylor we see that $|\phi(\zeta)| \sim |\zeta|$ near the origin, and so $e^{-\varphi} \sim |z|^{-2n}$ which is not locally integrable at 0. Also note that $\varphi(z) \leq 2n \log(4/\pi) + 1$ for $z \in K_{\mathbb{C}}$.

Fix a small δ and let g be a cut-off function on $\delta K_{\mathbb{C}}$. Applying the Hörmander theorem to $-\bar{\partial}g$ produces h such that $\bar{\partial}(h+g) = 0$. Call $f = h+g$ this holomorphic extension of g. Then $f(0) = 1$ and

$$\begin{aligned}\|f\|^2_{A^2(T_K)} &\leq 2(\|h\|^2_{L^2(T_K)} + \|g\|^2_{L^2(T_K)}) \\ &\leq 2(\|e^\varphi\|_{L^\infty} R^2 \delta^{-2} \|\bar{\partial}g\|^2_{L^2(e^{-\varphi})} + \|g\|^2_{L^2}).\end{aligned}$$

One can choose g appropriately so that $\|f\|^2_{A^2(T_K)} \leq \left(\frac{4}{\pi}\right)^{2n} e^{o(n)} (\operatorname{vol} K)^2$ as $\delta \to 0$. This gives the lower bound

$$\left(\frac{\pi}{4}\right)^{2n} \frac{e^{-o(n)}}{(\operatorname{vol} K)^2} \leq \mathcal{K}(0,0) \leq \frac{n!}{\pi^n} \frac{\operatorname{vol} K^\circ}{\operatorname{vol} K}.$$

One can remove the exponential factor with a "tensor power trick" to obtain

$$\left(\frac{\pi}{4}\right)^{2n} \leq \frac{n!}{\pi^n} \operatorname{vol} K^\circ \operatorname{vol} K$$

which gives the value $c = \left(\frac{\pi}{4}\right)^3$ in (2).

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