The Bourgain–Milman (via Hörmander's) theorem

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Reduce the problem to prove a lower bound for $\operatorname{vol} K \cdot \operatorname{vol} K^\circ$ to constructing an analytic function on $\mathbb{C}^n$ with good decay property.

By the Paley–Wiener theorem, given any $g \in L^2(K^\circ)$ its Fourier transform $f(w) = \int_{K^\circ} g(v) e^{-i\langle w,v\rangle} \mathrm{d}v$ extends to an entire function on $\mathbb{C}^n$. Applying Cauchy–Schwarz $|f(0)|^2 \leq \|g\|^2_{L^2(K^\circ)} \operatorname{vol} K^\circ$, and Plancherel $\|f\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^n \|g\|^2_{L^2(K^\circ)}$ we have the lower bound

$$\operatorname{vol} K^\circ \geq (2\pi)^n \frac{|f(0)|^2}{\|f\|^2_{L^2(\mathbb{R}^n)}}.$$

We want an entire function which $L^2(\mathbb{R}^n)$-norm is not too large compared with its value at the origin. We look for such a function in a Bergman space with Hörmander type weight, i.e. $L^2(\mathbb{C}^n, e^{-\varphi})$ where φ is plurisubharmonic.

Let T_K be the (horizontal) tube domain $\{x + iy : x \in \mathbb{R}^n, y \in K\}$ and consider the Bergman space $A^2(T_K) = \{\text{analytic functions on } T_K\} \cap L^2(T_K)$.

This is a Hilbert space with reproducing kernel, i.e. for any $f \in A^2(T_K)$ the evaluation at a point z can be written as $f(z) = \int_{T_K} \mathcal{K}(z,w) f(w) \mathrm{d}w$ where $\mathcal{K}$ is

$$\mathcal{K}(z,w) = \int_{\mathbb{R}^n} \frac{e^{i\langle z-\bar{w},v\rangle}}{\int_K e^{-2\langle x,v\rangle} \mathrm{d}x} \frac{\mathrm{d}v}{(2\pi)^n}.$$

An application of Cauchy–Schwarz gives

$$|f(0)|^2 = \left| \int_{T_K} \mathcal{K}(0,w) f(w) \mathrm{d}w \right|^2 \leq \int |\mathcal{K}(0,w)|^2 \int |f(w)|^2 = \mathcal{K}(0,0) \|f\|^2_{A^2(T_K)}$$

from which the lower bound follows

$$\frac{|f(0)|^2}{\|f\|^2_{A^2(T_K)}} \le \mathcal{K}(0,0) = \int_{\mathbb{R}^n} \frac{1}{\int_K e^{-2\langle x,v\rangle} \mathrm{d}x} \frac{\mathrm{d}v}{(2\pi)^n} \le \frac{n!}{\pi^n} \frac{\operatorname{vol} K^\circ}{\operatorname{vol} K}.$$

and the upper one using the convexity of $x \mapsto e^{-\langle x,v\rangle}$ and optimising in v.

Remark 1. Let $\|y\|_{K^\circ} = \inf\{\lambda > 0 \colon \lambda K^\circ \ni y\}$ be the Minkowski functional of K°. By convexity of $x \mapsto e^{-\langle x,v\rangle}$

$$\int_K e^{-2\langle x,y\rangle} \mathrm{d}x \ge 2^{-n} \operatorname{vol} K e^{\|-y\|_{K^\circ}}$$

Up to affine linear transformations, we can assume that K contains the ball $B(0,r)$. By the John's ellipsoid theorem $K \subset B(0,R)$ with $R/r \le \sqrt{n}$. For any $t \in K^\circ$, the Hermitian product $z \mapsto \langle z,t\rangle$ maps T_K in the strip $S = \{\zeta \in \mathbb{C} : \ |\Im(\zeta)| < 1\}$, while the conformal map

$$\phi(\zeta) = \frac{4}{\pi} \frac{e^{\frac{\pi}{2}\zeta} - 1}{e^{\frac{\pi}{2}\zeta} + 1}$$

maps the strip S to the disk $D(0, \frac{4}{\pi})$. Consider the set

$$K_{\mathbb{C}} := \{z \in \mathbb{C}^n \ : \ |\langle z,t\rangle| \le 1, \forall t \in K^\circ\} \subset T_K.$$

Note that $K_{\mathbb{C}}$ contains $\frac{1}{\sqrt{2}}(K + iK)$. It is enough to construct an analytic function inside $K_{\mathbb{C}}$. For this purpose we will use the Hörmander's theorem.

Definition 1. A function $\varphi \colon \Omega \subset \mathbb{C}^n \to \mathbb{R}$ is *strictly plurisubharmonic* if there exists $\tau > 0$ such that

$$\langle H(z)w, w\rangle \ge \tau |w|^2, \quad \forall w \in \mathbb{C}^n, \forall z \in \Omega$$

where H is the Hermitian matrix $H = \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}\right)_{i,j=1}^n$.

Theorem 1 (Hörmander)**.** *Let $\Omega \subset \mathbb{C}^n$ be an open, pseudoconvex domain, and let $\varphi \colon \Omega \to \mathbb{R}$ be strictly plurisubharmonic for a $\tau > 0$. For any $(0,1)$-form ω on Ω with $\bar{\partial}\omega = 0$, there exists a solution h of $\bar{\partial} h = \omega$ in Ω satisfying*

$$\int_\Omega |h|^2 e^{-\varphi} \mathrm{d}z \le \tau^{-1} \int_\Omega |\omega|^2 e^{-\varphi} \mathrm{d}z.$$

We take the plurisubharmonic function φ on a shrunk version of $K_{\mathbb{C}}$:

$$\varphi(z) = \frac{|\Im(z)|^2}{R^2} + \log \sup_{t \in K^\circ} |\phi(\langle z, t\rangle)|^{2n}.$$

The first term enforces the strict plurisubharmonicity on any ball of radius $\delta < R$ with $\tau = \delta^2/R^2$. The second term ensures that the function h promised by the theorem will vanish at 0, as soon as $\int |\omega|^2 e^{-\varphi}$ is finite. Indeed, since $\phi(0) = 0$ and $\phi'(0) = 1$, using Taylor we see that $|\phi(\zeta)| \sim |\zeta|$ near the origin, and so $e^{-\varphi} \sim |z|^{-2n}$ which is not locally integrable at 0. Also note that $\varphi(z) \leq 2n \log(4/\pi) + 1$ for $z \in K_{\mathbb{C}}$.

Fix a small δ and let g be a cut-off function on $\delta K_{\mathbb{C}}$. Applying the Hörmander theorem to $-\bar{\partial} g$ produces h such that $\bar{\partial}(h+g) = 0$. Call $f = h+g$ this holomorphic extension of g. Then $f(0) = 1$ and

$$\begin{aligned} \|f\|^2_{A^2(T_K)} &\leq 2(\|h\|^2_{L^2(T_K)} + \|g\|^2_{L^2(T_K)}) \\ &\leq 2(\|e^{\varphi}\|_{L^\infty} R^2 \delta^{-2} \|\bar{\partial} g\|^2_{L^2(e^{-\varphi})} + \|g\|^2_{L^2}). \end{aligned}$$

One can choose g appropriately so that as $\delta \to 0$ $\|f\|^2_{A^2(T_K)} \leq \left(\frac{4}{\pi}\right)^{2n} e^{o(n)} (\operatorname{vol} K)^2$. Then

$$\left(\frac{\pi}{4}\right)^{2n} \frac{e^{-o(n)}}{\operatorname{vol}(K)^2} \leq \mathcal{K}(0,0) \leq \frac{n!}{\pi^n} \frac{\operatorname{vol} K^\circ}{\operatorname{vol} K}.$$

After removing the exponential with a "tensor power trick", obtain

$$\left(\frac{\pi}{4}\right)^{2n} \leq \frac{n!}{\pi^n} \operatorname{vol} K^\circ \operatorname{vol} K.$$

which gives the value $c = \left(\frac{\pi}{4}\right)^3$ if one multiplies by the Mahler volume of the hypercube C_n in $\mathbb{R}^n$:

$$\operatorname{vol} C_n^\circ \operatorname{vol} C_n = \frac{4^n}{n!}.$$

A The John ellipsoid

Following §2.4 in the lecture notes "Dyadic analysis and weights" by Tuomas Hytönen, 2017.

Theorem 2 (John ellipsoid theorem)**.** *Let $K \subset \mathbb{R}^d$ be a compact convex symmetric set. Then there exists a closed ellipsoid E centered at the origin such that $E \subset K \subset \sqrt{d} E$.*

Lemma 3. *Let $K \subset \mathbb{R}^d$ be a compact non-empty set. Among all closed ellipsoid $E \subset K$ centered at the origin there is one of maximal measure.*

Idea of the proof. The ellipsoid E is the image of a unit ball B via a matrix $A \in \mathrm{Mat}(\mathbb{R}^d) \cong \mathbb{R}^{d\times d}$. The condition $AB \subset K$ forces the set of such matrices to be *compact*. The function $|E| = \det(A)$ is continuous, and so it has a maximum on a compact set. □

Let $B \subset \mathbb{R}^d$ the unit ball. Given a point $p = (p_1, \vec{0})$, there is an ellipsoid E in the convex hull of $B \cup \{p, -p\}$.

Lemma 4. *Let $K \subset \mathbb{R}^d$ be a convex, symmetric body. Suppose that the unit ball B is the maximum ellipsoid among all origin-centered ellipsoid $E \subset K$. Then $K \subset \sqrt{d}B$.*

Idea of the proof. Take a point $p \in K$, then $-p \in K$. There is an ellipsoid E in the K, which contains the convex hull of $B \cup \{p, -p\}$. The family of such ellipsoids is given by E_t for $t \in (0,1)$. Since $B = E_1$ has the maximal measure among the ellipsoid, $\frac{\mathrm{d}}{\mathrm{d}t}|E_t|^2\restriction_{t=1} \geq 0$ and by scaling property

$$\frac{\mathrm{d}}{\mathrm{d}t}|E_t|^2\restriction_{t=1} = |B|^2(d - |p|^2) \geq 0$$

which gives $|p| \leq \sqrt{d}|B|$. □

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