Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations

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after

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We consider the cubic one-dimensional defocusing nonlinear Schrödinger equation (NLS):

$$\begin{cases} -iu_t + \triangle u = |u|^2 u \\ u(0,x) = u_0(x) \qquad u_0 \in H^s_x(\mathbb{R}). \end{cases} \tag{NLS}$$

We consider solutions in the space $C^0([0,T], H^s(\mathbb{R}))$ that are fixed points of the Duhamel map:

$$u(t) = U(t)u_0 + i\int_0^t U(t-s)|u|^2 u(s)\,\mathrm{d}s,$$

with $U(t) = e^{-it\triangle}$ defined as $U(t)u_0 := \mathcal{F}^{-1}(e^{it\xi^2}\widehat{u_0})$, where $\mathcal{F}$ is the Fourier Transform.

1 Well-posedness

We will say that our Initial Value Problem is

locally well-posed in H^s if for every ball B_R of radius $R > 0$ exists a time $T = T(R) > 0$ such that the solution operator

$$S\colon B_R = \{u_0 \in H^s : \|u_0\|_{H^s} < R\} \to C^0([0,T), H^s(\mathbb{R}))$$

is uniformly continuous.

globally well-posed if T can be arbitrarily large independent of R.

1.1 Previous result

Theorem 1 (Tsutsumi, 1987)**.** *The defocusing* (NLS) *is globally well-posed in $H^s(\mathbb{R})$ for $s \geq 0$.*

The main result of this talk is:

Theorem 2. *The cubic one-dimensional defocusing* (NLS) *is* not *locally well-posed in $H^s(\mathbb{R})$ for $s < 0$.*

2 Symmetries

Let $u(t,x)$ be a solution of (NLS), then are solutions as well:

Symmetry		Invariant norm
Scaling	$u^\lambda(t,x) = \frac{1}{\lambda}u\left(\frac{t}{\lambda^2},\frac{x}{\lambda}\right)$	$\dot{H}_x^{-1/2}$
Galilean	$G_v(u)(t,x) = e^{itv^2}e^{ixv}u(t,x+2vt)$	L_x^2

$$H^s(\mathbb{R}) := \left\{u \in \mathcal{S}'(\mathbb{R}) | (1+|\xi|^2)^{s/2}\widehat{u}(\xi) \in L^2(\mathbb{R})\right\}$$

2.1 Free Schrödinger and pseudo-conformal transformation

Consider the Free Schrödinger Equation for forward time

$$\begin{cases} -iu_t + \triangle u = 0 \quad t > 0 \\ u(0,x) = u_0(x). \end{cases} \tag{FSE}$$

Let $u(t,x)$ be a solution of (FSE). We introduce the *pseudo-conformal* transformation by setting:

$$v(s,y) := \mathrm{pc}(u) = s^{-1/2}\exp\left(iy^2/4s\right)\, u\left(\frac{1}{s}-1,\frac{y}{s}\right).$$

This transformation is associated to the following change of variables:

$$(y,s) = \left(\frac{x}{1+t},\frac{1}{1+t}\right),\ (x,t) = \left(\frac{y}{s},\frac{1}{s}-1\right).$$

The transformed $v(s,y) := \mathrm{pc}(u)$ solves the backwards Free Schrödinger for time :

$$\begin{cases} iv_s + \triangle v = 0 \quad s \in (0,1] \\ v(1,y) = v_1(y). \end{cases} \tag{bFSE}$$

We now can solve this problem for any time $s < 1$, and in particular the solution extends continuously to $s = 0$:

$$v(s,y) \xrightarrow[L^2]{s\to 0^+} v(0,y) =: \varphi(y).$$

Using the inverse transformation $u = \mathrm{pc}^{-1}(v)$ we see that:

$$u(t,x) \overset{t\to+\infty}{\approx} (1+t)^{-1/2}\exp\left(-ix^2/4(1+t)\right)\varphi\left(\frac{x}{1+t}\right).$$

3 Backwards Nonlinear Schrödinger equation

Applying the pc transformation to solutions of (NLS) we obtain a new initial value problem for $v(s,y) := \mathrm{pc}(u)$:

$$\begin{cases} iv_s + \triangle v = s^{-1}|v|^2 v \quad s \in (0,1] \\ v(1,y) = v_1(y) \end{cases} \tag{bNLS}$$

Dropping $\partial_{yy}v$, we can explicitly solve the ODE:

$$\begin{cases} iv_s^{[w]} = s^{-1}|v^{[w]}|^2 v^{[w]} \\ v(1,y) = w(y) \end{cases} \tag{ODE}$$

finding the solution

$$v^{[w]}(s,y) = w(y)\exp(-i|w(y)|^2\log(s)).$$

We notice that as $s \to 0^+$:

$$|v_s^{[w]}| \sim s^{-1}, \quad |\triangle v^{[w]}| \sim (\log s)^2, \quad \left|s^{-1}|v^{[w]}|^2 v^{[w]}\right| \sim s^{-1},$$

and since $|\log s| \ll s^{-1}$, neglecting the term $\triangle v^{[w]}$ is reasonable and $v^{[w]}$ turns out to be "close" to a solution of (bNLS).

To make this precise we introduce a weighted Sobolev norm:

$$\|v\|_{\mathbf{H}_y^{k,k}} := \sum_{\substack{\alpha,\beta\geq 0:\\ \alpha+\beta\leq k}} \left\|y^\alpha \partial_y^\beta v\right\|_{L_y^2}.$$

Consider the ball $B_\epsilon = \{\|u\|_{\mathbf{H}^{k+2,k+2}} < \epsilon\}$, where $\epsilon \ll 1$. Let be $k \in \mathbb{N}, k \geq 5$.

Lemma 1. *For all $w \in B_\epsilon$ exists $v_1 \in \mathbf{H}^{k,k}$ such that the unique solution $v^{\langle w\rangle}$ to (bNLS) in $\mathbf{H}^{k,k}$ with initial data v_1 satisfies*

$$\left\|v^{\langle w\rangle}(s) - v^{[w]}(s)\right\|_{\mathbf{H}^{k,k}} \lesssim \epsilon\, s\,(1+|\log s|)^C \qquad \textit{for all } s \in (0,1]$$

Furthermore, the map from B_ϵ to $C^0((0,1],\mathbf{H}^{k,k})$, $w \mapsto v^{\langle w\rangle}$ is Lipschitz up to the end time $s = 0$.

3.1 Decoherence in L^2

We fix w (e.g. $w(y) = \epsilon e^{-y^2}$, so $w \approx \epsilon$). For $a \in \mathbb{R}$ we can take the initial value $aw(y)$. We have this:

Lemma 2 (decoherence property)**.** *If $a, a' \lesssim 1$ and $a \neq a'$, then*

$$\limsup_{s\to 0^+} \left\|v^{[aw]}(s) - v^{[a'w]}(s)\right\|_{L_y^2} \gtrsim \left(|a| + |a'|\right)\|w\|_{L_y^2}.$$

The last two Lemmas together imply

$$\limsup_{s\to 0^+} \left\|v^{\langle aw\rangle}(s) - v^{\langle a'w\rangle}(s)\right\|_{L_y^2} \gtrsim \epsilon.$$

Using the inverse pc we may construct a family of solutions $u^{\langle aw\rangle} = \mathrm{pc}^{-1}(v^{\langle aw\rangle})$, and $u^{[aw]} = \mathrm{pc}^{-1}(v^{[aw]})$ such that:

a) $\left\|u^{\langle aw\rangle}(0)\right\|_{H^k} \lesssim \left\|v^{\langle aw\rangle}(1)\right\|_{\mathbf{H}^{k,k}} \lesssim \epsilon$;

b) $\left\|u^{\langle aw\rangle}(0) - u^{\langle a'w\rangle}(0)\right\|_{H^k} \lesssim \left\|v^{\langle aw\rangle}(1) - v^{\langle a'w\rangle}(1)\right\|_{\mathbf{H}^{k,k}} \lesssim \epsilon|a-a'|$;

c) $\left\|u^{\langle aw\rangle}(t) - u^{[a'w]}(t)\right\|_{H^k} \lesssim \epsilon\,(1+t)^{-1}\,(1+|\log(1+t)|)^C$;

d) $\limsup_{t\to+\infty}\left\|u^{\langle aw\rangle}(t) - u^{\langle a'w\rangle}(t)\right\|_{L^2} = \limsup_{s\to 0^+}\left\|v^{\langle aw\rangle}(s) - v^{\langle a'w\rangle}(s)\right\|_{L^2} \gtrsim \epsilon.$

4 Ill-posedness

To disprove uniform continuity of $S\colon B_R \subset H^s(\mathbb{R}) \to C^0([0,T), H^s(\mathbb{R}))$, for $s \in \left(-\frac{1}{2}, 0\right)$, we will show that for any $T > 0$ there exists $\epsilon > 0$ such that for any $\delta > 0$ there are two solutions ϕ, ϕ' such that:

- $\phi(0), \phi'(0) \in B_R$,
- $\|\phi(0) - \phi'(0)\|_{H^s} \lesssim \delta$,
- $\sup_{[0,T)} \|\phi(t) - \phi'(t)\|_{H^s} \gtrsim \epsilon$.

We take $N \gg 1$ and consider:

$$\phi^{\langle a\rangle}(t,x) := G_N\left(u^\lambda\right)(t,x) = \frac{1}{\lambda} e^{itN^2} e^{ixN} u^{\langle aw\rangle}\left(\frac{t}{\lambda^2}, \frac{(x+2Nt)}{\lambda}\right).$$

We need a lemma to control the H^s norm of $\phi^{\langle a\rangle}$ by the H^k norm from above and by the L^2 norm from below:

Lemma 3. *Let be $k \in \mathbb{N}$, $u \in H^k$, $N \geq 1$, $\lambda \in \mathbb{R}^+$, and let be $\phi^{\langle a\rangle}$ defined as above. Then*

i) For $s < 0$, $k \geq |s|$, whenever $1 \leq \lambda N^{1+(s/k)}$, we have:

$$\|\phi\|_{H^s} \lesssim \lambda^{-\frac{1}{2}} N^s \|u\|_{H^k},$$

ii) For every $u \in H^k$ exists a constant $C_u < \infty$ such that when $\lambda N \geq C_u$ we have:

$$\|\phi\|_{H^s} \gtrsim \lambda^{-\frac{1}{2}} N^s \|u\|_{L^2}$$

Taking $a, a' \lesssim 1$, consider $\phi^{\langle a\rangle}$ and $\phi^{\langle a'\rangle}$. Applying the Lemma and choosing $\lambda = N^{2s}$ these two solutions disprove the uniform continuity of S, in fact:

a) $\left\|\phi^{\langle a\rangle}(0)\right\|_{H^s} \lesssim \left\|u^{\langle aw\rangle}(0)\right\|_{H^k} \lesssim \epsilon$;

b) $\left\|\phi^{\langle a\rangle}(0) - \phi^{\langle a'\rangle}(0)\right\|_{H^s} \lesssim \epsilon|a-a'|$;

d) $\limsup_{t\to+\infty} \left\|\phi^{\langle a\rangle}(t) - \phi^{\langle a'\rangle}(t)\right\|_{H^s} = \limsup_{t\to+\infty} \left\|u^{\langle aw\rangle}(t) - u^{\langle a'w\rangle}(t)\right\|_{L^2} \gtrsim \epsilon$.

Thus there exists t_0 such that $\left\|u^{\langle aw\rangle}(t_0) - u^{\langle a'w\rangle}(t_0)\right\|_{L^2} \gtrsim \epsilon$. Let $t = \lambda^2 t_0$. Making N arbitrarily large, $\lambda = N^{2s}$ becomes arbitrarily small, so $t \in [0,T)$ for any time $T > 0$.