Modulation spaces, Wiener Amalgam spaces, and Brownian motions

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after

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Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. We consider a complex-valued Brownian motion

$$\beta\colon [0,1] \times \Omega \to \mathbb{C}$$

such that

(i) $\beta(0,\omega) = 0$ for almost every $\omega \in \Omega$,

(ii) $\beta(t,\omega)$ has independent increments and $\beta(t) - \beta(s) \sim \mathcal{N}(0, t-s)$, for all $0 \le s \le t < 1$.

There is a version of β such that $\mathbb{P}(t \mapsto \beta(t,\omega)$ is continuous$) = 1$.

1 Previous results

Theorem 1. *The Brownian motion $\beta(t)$ belongs almost surely to the Sobolev spaces $H^s_{loc}, W^{s,p}_{loc}$ if and only if $s < \frac{1}{2}$, regardless of $p \in [1,\infty]$.*

Theorem 2. *The Brownian motion $\beta(t)$ belongs almost surely to the Besov spaces $(B^s_{p,q})_{loc}$ if and only if $s < \frac{1}{2}$, and $p, q \in [1,\infty]$, or if $s = \frac{1}{2}$ for $1 \le p < \infty$ and $q = \infty$.*

Let $1 \le p, q \le \infty$. We consider the modulation spaces on the torus $M^{p,q}_s(\mathbb{T})$. One of the two main results of this talk is:

Theorem 3. *The mean zero Brownian motion $u(t)$ belongs a.s. to $M^{p,q}_s(\mathbb{T})$ if and only if*

(a) $q < \infty$ *and* $(s-1)q < -1$.

(b) $q = \infty$ *and* $s < 1$.

2 Function spaces

Definition 1. Let be $\langle \cdot \rangle^s := (1 + |\cdot|^2)^{\frac{s}{2}}$. Then we recall the following function spaces:

Sobolev spaces $(p = 2)$ $\quad H^s(\mathbb{R}) = \left\{ f \in \mathcal{S}'(\mathbb{R}) \ : \ \langle \xi \rangle^s \widehat{f}(\xi) \in L^2(\mathbb{R}) \right\}$

Sobolev spaces $\quad W^{s,p}(\mathbb{R}) = \left\{ f \in \mathcal{S}'(\mathbb{R}) \ : \ \left(\langle \xi \rangle^s \widehat{f}(\xi) \right)^{\check{}} \in L^p_x(\mathbb{R}) \right\}$

Fourier-Lebesgue spaces $\quad \mathcal{F}L^{s,p}(\mathbb{R}) = \left\{ f \in \mathcal{S}'(\mathbb{R}) \ : \ \langle \xi \rangle^s \widehat{f}(\xi) \in L^p_\xi(\mathbb{R}) \right\}$

Consider a window function $g \in \mathcal{S}(\mathbb{R})$.

Modulation spaces $\quad M^{p,q}_s(\mathbb{R}) = \left\{ f \in \mathcal{S}'(\mathbb{R}) \ : \ \langle \xi \rangle^s V_g f(x, \xi) \in L^p_x(\mathbb{R}) L^q_\xi(\mathbb{R}) \right\}$

Wiener Amalgam spaces $\quad \mathrm{W}^{p,q}_s(\mathbb{R}) = \left\{ f \in \mathcal{S}'(\mathbb{R}) \ : \ \langle \xi \rangle^s V_g f(x, \xi) \in L^q_\xi(\mathbb{R}) L^p_x(\mathbb{R}) \right\}$

Besov spaces $\quad \|f\|_{B^s_{p,q}(\mathbb{R})} = \|\|(\langle \xi \rangle^s \varphi_j(\xi) \widehat{f}(\xi))^{\check{}}\|_{L^p_x(\mathbb{R})}\|_{\ell^q_j(\mathbb{N}) < \infty}.$

Consider a bump function φ_0, and define $\varphi_j(x) = \varphi(2^j x) - \varphi(2^{j-1} x)$, for $j \in \mathbb{N}$, such that $\sum_j \varphi_j = 1$.

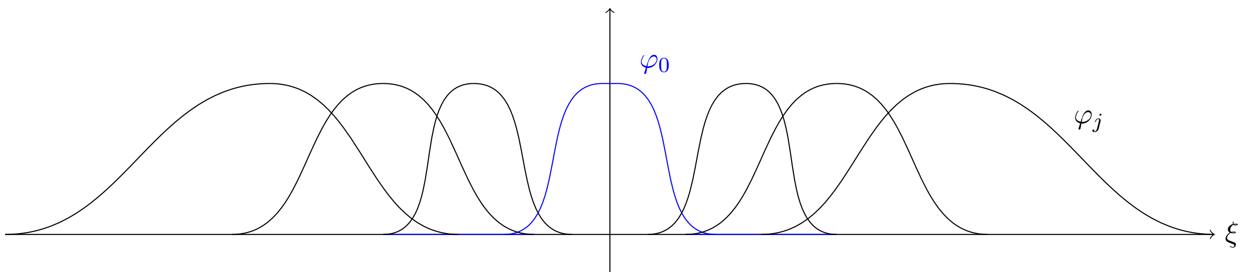

Figure 1: Plot of the functions φ_j for the Littlewood-Paley decomposition.

3 Brownian motions

Let B_t be a Brownian motion on $\mathbb{R}_+$. Consider an isometry

$$\begin{aligned} L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \, \mathrm{d}t) &\xrightarrow{G} \mathcal{G}(\Omega, \mathcal{A}, \mathbb{P}) \\ f &\mapsto \mathcal{N}(0, \|f\|_2^2) \end{aligned}$$

where $\mathcal{G}(\Omega, \mathcal{A}, \mathbb{P})$ is a space of centered Gaussian random variables; $\mathcal{B}(\mathbb{R}_+)$ indicates the class of Borel set of $\mathbb{R}_+$, and $\mathrm{d}t$ is the Lebesgue measure. Then

$$B(t) := G(\mathbb{1}_{[0,t]}) = \mathcal{N}\left(0, \int_0^\infty \mathbb{1}_{[0,t]}(s) \, \mathrm{d}s\right) = \mathcal{N}(0, t).$$

3.1 Brownian loop and Fourier analytic representation

Let be B_t a classic Brownian motion on $\mathbb{R}_+$.

Consider

$$\beta(t) := B(t) - \frac{t}{2\pi} B(2\pi), \qquad \text{for } t \in [0, 2\pi).$$

By the invariance of B_t, this is a periodic function. For studying the local regularity it is enough to consider the *mean zero* loop, that we indicate with $u(t)$, such that $\int_0^{2\pi} u(t) \, \mathrm{d}t = 0$. We can express u via a Fourier-Wiener series.

Since β is periodic, consider the isometry

$$\begin{aligned} L^2([0,2\pi]) &\xrightarrow{T} \mathcal{G}(\Omega,\mathcal{A},\mathbb{P}) \\ f &\mapsto \int_0^{2\pi} \overline{f}(s)\,\mathrm{d}\beta(s) = \int_0^{2\pi} \overline{f_0}(s)\,\mathrm{d}B(s) \sim \mathcal{N}(0,\sigma^2) \end{aligned}$$

where f_0 is the mean zero part of f, $f_0(t) := f(t) - \fint_0^{2\pi} f(s)\,\mathrm{d}s$ and $\sigma^2 = 2\|f\|_2^2$[1]

Then $\beta(t) = T(\mathbb{1}_{[0,t]})$. Let be $\{e_n\}_{n\in\mathbb{Z}}$ an orthonormal basis of $L^2([0,2\pi])$. We can expand any function as Fourier series, so

$$\beta(t) = T(\mathbb{1}_{[0,t]}) = T(\sum c_n e_n) = \sum_{n\in\mathbb{Z}} c_n T(e_n) = \sum_{n\in\mathbb{Z}} c_n(t)\, g_n(\omega)$$

where g_n is a centered Gaussian random variable, since $T(e_n) \sim \mathcal{N}(0,2)$. Subtracting the average, since $c_n(t) = \langle \mathbb{1}_{[0,t]}, e_n\rangle_2 = \frac{e^{int}}{\sqrt{2\pi}in}$, we obtain (up to constant) the following representation of the periodic, mean zero Brownian loop on $[0,2\pi)$:

$$\boxed{u(t,\omega) = \sum_{n\neq 0} \frac{g_n(\omega)}{n}\, e^{int}}\,.$$

4 Regularity of Brownian motion

We study local-in-time regularity of the sample paths $t \mapsto \beta(t)$. Localized version of the spaces coincide with equivalent norms

$$M_s^{p,q}(\mathbb{T}) = \mathbb{W}_s^{p,q}(\mathbb{T}) = \mathcal{F}L^{s,q}(\mathbb{T}).$$

Thus in the proof we can use the norm of the Fourier-Lebesgue space

$$\|u\|_{\mathcal{F}L^{s,q}(\mathbb{T})} = \|\langle k\rangle^s\, \widehat{u}(k)\|_{\ell_k^q(\mathbb{Z})}.$$

Proof of Theorem 3 for $q<\infty$. Denote with $\mathbb{E}$ the expectation, we have

$$\mathbb{E}\left[\|u\|^q_{\mathcal{F}L^{s,q}(\mathbb{T})}\right] = \sum_{n\neq 0} \langle n\rangle^{sq}\, |n|^{-q}\mathbb{E}\left[|g_n|^q\right] \sim \sum_{n\neq 0} \langle n\rangle^{(s-1)q} < \infty$$

if and only if $(s-1)q < -1$.

On the other hand

$$\begin{aligned} \|u\|^q_{\mathcal{F}L^{s,q}(\mathbb{T})} &= \sum_{n\neq 0} \langle n\rangle^{sq}\, |n|^{-q} |g_n(\omega)|^q \sim \sum_{j=0}^{\infty} \sum_{|n|\sim 2^j} \langle n\rangle^{(s-1)q}\, |g_n(\omega)|^q \\ &\geq \sum_{j=0}^{\infty} \sum_{|n|\sim 2^j} \langle n\rangle^{-1}\, |g_n(\omega)|^q \sim \sum_{j=0}^{\infty} X_j^{(q)}(\omega) = \infty, \text{ a.s.} \end{aligned}$$

where $X_j = 2^{-j} \sum_{|n|\sim 2^j} |g_n(\omega)|^q$.

□

[1] the 2 in front comes from the fact that we are considering the complex-valued Brownian motion.

5 Abstract Wiener Spaces

Let $H = \dot{H}^1(\mathbb{T})$ be a Hilbert space with the norm $\|u\|_H = \sum_{n\in\mathbb{Z}} |n|^2 |\widehat{u}(n)|^2$. Let be

$$\mathcal{F} = \{\text{finite rank projections on } H\} \longleftrightarrow \{\text{finite dimensional subspace of } H\}\,.$$

A *cylinder* set of H is $E = \{u \in H \,:\, Pu \in A\}$, where $P \in \mathcal{F}$, A is a Borel subset of $P(H)$. We can define a Gaussian measure on $\mathcal{R} = \{\text{cylinder set of } H\}$

$$\mu(E) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_A e^{-\frac{1}{2}\|u\|_H^2}\,\mathrm{d}u$$

where $d = \dim P(H)$, and $\mathrm{d}u$ is the Lebesgue measure on $P(H)$.

Definition 2. A seminorm $[\![\cdot]\!]$ on H is *measurable* if for every $\varepsilon > 0$ exists $P_0 \in \mathcal{F}$ such that

$$\mu\left(u \,:\, [\![Pu]\!] > \varepsilon\right) < \varepsilon \qquad \forall P \perp P_0,\ P \in \mathcal{F}.$$

Remark 1. The seminorm $[\![\cdot]\!]$ is weaker then $\|\cdot\|_H$.

Theorem 4. *The seminorms* $\|\cdot\|_{M_s^{p,q}(\mathbb{T})}$, $\|\cdot\|_{\mathbb{W}_s^{p,q}(\mathbb{T})}$, $\|\cdot\|_{\mathcal{F}L^{s,q}(\mathbb{T})}$, *are measurable on* H *for* $(s-1)q < -1$.

Corollary 1. *Let* μ *be the mean zero Wiener measure on the torus* $\mathbb{T}$. *Then the spaces* $(M_s^{p,q}(\mathbb{T}), \mu)$, $(\mathbb{W}_s^{p,q}(\mathbb{T}), \mu)$ *and* $(\mathcal{F}L^{s,q}(\mathbb{T}), \mu)$ *are abstract Wiener space for* $(s-1)q < -1$.

As a consequence of the Fernique theorem

Theorem 5 (Fernique)**.** *Let* (B, μ) *be an abstract Wiener space. Then there exists* $c' > 0$ *such that*

$$\mu(\|u\|_B \geq K) \leq e^{-c'K^2},$$

for sufficiently large $K > 0$.

we obtain large deviation estimates for the time-frequency spaces

Theorem 6. *If* $(s-1)q < -1$ *there exists* $c > 0$ *such that for (sufficiently large)* $K > 0$*:*

$$\mu\left(\|u(\omega)\|_{M_s^{p,q}(\mathbb{T})} > K\right) < e^{-cK^2}.$$

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