Endpoint Strichartz Estimates

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after

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We consider the homogeneous Schrödinger equation in $\mathbb{R}^d$:

$$\begin{cases} i\partial_t u - \triangle u = 0 \\ u(0,x) = u_0(x)\,, \quad u_0 \in \mathcal{S}(\mathbb{R}^d). \end{cases} \tag{1}$$

The solution is given by

$$u(t,x) = e^{-it\triangle}u_0 := (e^{it|\xi|^2}\widehat{u_0}(\xi))\check{},$$

where $\widehat{u_0}$ and $(u_0)\check{}$ are the Fourier Transform and the Inverse Fourier Transform on $\mathbb{R}^d$.

Scaling If u is a solution of (1) with initial data u_0, then $u_\lambda(t,x) = u(\lambda^2 t, \lambda x)$ is a solution with initial data $(u_0)_\lambda(x) = u_0(\lambda x)$.

1 Restriction theory

Look closer at the solution of Equation (1):

$$u(t,x) = e^{-it\triangle}u_0 = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|^2)}\widehat{u_0}(\xi)\,\mathrm{d}\xi.$$

We interpret the above display equality as an inverse space-time ($\mathbb{R}^{d+1}$) Fourier Transform:

$$u(t,x) = \mathcal{F}^{-1}(v(\tau,\xi)) = \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^{d+1}} e^{i(t,x)\cdot(\tau,\xi)}v(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi,$$

from which:

$$v(\tau,\xi) = 2\pi\,\widehat{u_0}(\xi)\delta(\tau - |\xi|^2),$$

where $\delta(\tau - |\xi|^2)$ is the measure on the paraboloid $\Sigma = \{(\tau,\xi) \in \mathbb{R}^{d+1}, \tau = |\xi|^2\}$.

Definition 1. Let $\mathcal{M} \subset \mathbb{R}^{d+1}$ be a d-dimensional manifold and μ a smooth measure supported on it. We define the following operators

Restriction operator	*Extension operator*
$\mathcal{R}\colon L^p(\mathbb{R}^{d+1}) \to L^2(\mathcal{M},\mu)$	$\mathcal{R}^\star\colon L^2(\mathcal{M},\mu) \to L^{p'}(\mathbb{R}^{d+1})$
$F \mapsto (\mathcal{F}F)_{\restriction\mathcal{M}}$	$g \mapsto \mathcal{F}^{-1}(g\,\mu)$

Thus, the solution of the Schrödinger equation (1) is given by applying the extension operator $\mathcal{R}^\star$ to the function $\widehat{u_0}$ when $\mathcal{M}$ is the paraboloid $\Sigma = \{(\tau,\xi) \in \mathbb{R}^{d+1}, \tau = |\xi|^2\}$ with the measure $\delta(\tau - |\xi|^2)$.

Theorem 1 (Tomas-Stein)**.** *Let $\mathcal{M} \subset \mathbb{R}^{d+1}$ a* compact[1] *d-dimensional manifold with non vanishing Gaussian curvature, and $f \in L^p(\mathbb{R}^{d+1})$, then*

$$\|\mathcal{R}f\|_{L^2(\mathcal{M})} \lesssim \|f\|_{L^p(\mathbb{R}^{d+1})} \quad \textit{holds for} \quad 1 \le p \le \frac{2(d+2)}{d+4}.$$

The dual statement for the extension operator reads:

Theorem 2 (Dual Tomas-Stein)**.** *Let $\mathcal{M} \subset \mathbb{R}^{d+1}$ a* compact *d-dimensional manifold with non vanishing Gaussian curvature, and $g \in L^2(\mathcal{M})$, then*

$$\|\mathcal{R}^\star g\|_{L^{p'}(\mathbb{R}^{d+1})} \lesssim \|g\|_{L^2(\mathcal{M})} \quad \textit{holds for} \quad p' \ge 2 + \frac{4}{d}. \tag{2}$$

Remark 1. The operator $e^{-it\triangle}$ is the composition of $\mathcal{R}^\star$ with the spatial Fourier Transform.

Remark 2. The Tomas-Stein inequality (2) holds on *compact* hypersurface. We can get rid of this assumption via rescaling. Consider $u_0 \in L^2(\mathbb{R}^d)$ such that

$$\operatorname{supp}(\widehat{u_0}) \subseteq \mathbb{B}^d_1 = \{\xi \in \mathbb{R}^d \,:\, |\xi| \le 1\}.$$

Rescaling u_0 with $\lambda > 0$, the Fourier Transform changes with the dual scaling:

$$(u_0)_\lambda(x) = u_0(\lambda x) \quad \Rightarrow \quad \widehat{(u_0)_\lambda}(\xi) = \lambda^{-d}\widehat{u_0}(\xi/\lambda) = \widehat{u_0}^\lambda(\xi),$$

then $\widehat{u_0}^\lambda$ is supported on $\mathbb{B}^d_\lambda = \{\xi \in \mathbb{R}^d \,:\, |\xi| \le \lambda\}$. The rescaled extension inequality (2):

$$\left\|\mathcal{R}^\star\widehat{u_0}^\lambda\right\|_{L^{p'}(\mathbb{R}^{d+1})} = \lambda^{-\frac{d+2}{p'}} \|\mathcal{R}^\star\widehat{u_0}\|_{L^{p'}(\mathbb{R}^{d+1})} \le C\lambda^{-\frac{d}{2}} \|\widehat{u_0}\|_{L^2(\mathcal{M})} = \left\|\widehat{u_0}^\lambda\right\|_{L^2(\mathcal{M})}$$

holds with the constant $C_\lambda = C\lambda^{-\frac{d}{2}+\frac{d+2}{p'}}$. In particular, for the value $p' = 2 + \frac{4}{d}$ we have $C_\lambda = C$ for every $\lambda > 0$. From Theorem 2, letting $\lambda \to \infty$ we obtain the bound for the whole paraboloid Σ. Since functions with compactly supported Fourier Transform are dense in L^2, with a limiting argument we obtain the extension inequality for all initial data in L^2.

2 Strichartz estimates for Schrödinger equation

Restriction theory gives estimates in time and space only on isotropic Lebesgue space (on $L^q_t(\mathbb{R})L^p_x(\mathbb{R}^d)$ when $q = p$). The paraboloid is invariant under anisotropic scaling

$$(x,t) \mapsto (\lambda x, \lambda^2 t)$$

so it is reasonable to study restriction and extension on anisotropic spaces ($q \ne p$):

$$\|e^{-it\triangle}u_0\|_{L^q_tL^p_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)}\,. \tag{3}$$

Proving this inequality is equivalent to showing either of the following:

- $T := e^{-it\triangle} \quad : L^2(\mathbb{R}^d) \longrightarrow L^q_tL^p_x(\mathbb{R}\times\mathbb{R}^d)$ is bounded,
- $T^\star := (e^{-it\triangle})^\star \,:\, L^{q'}_tL^{p'}_x(\mathbb{R}\times\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is bounded.

[1] or $\mathcal{M}$ is a hypersurface with a compactly supported measure μ.

The composition $TT^\star$:

$$e^{-it\triangle}(e^{-is\triangle})^\star \colon L_t^{q'}L_x^{p'}(\mathbb{R}\times\mathbb{R}^d) \to L_t^q L_x^p(\mathbb{R}\times\mathbb{R}^d) \quad \text{is a bounded operator.}$$

We will prove the last bound for $TT^\star$ and, by Hölder and duality, the previous follow.

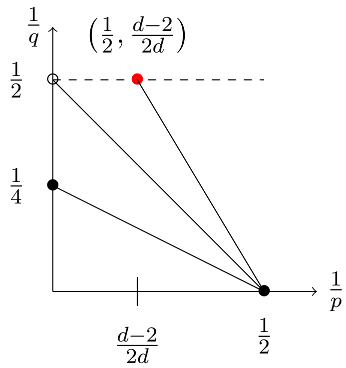
Theorem 3 (Nonendpoint estimates)**.** *The operator $TT^\star$ is given by $u \mapsto \int_{-\infty}^{+\infty} e^{-i(t-s)\triangle} u \,\mathrm{d}s$ and the following inequality:*

$$\left\| \int_{-\infty}^{\infty} e^{-i(t-s)\triangle} F(s)\,\mathrm{d}s \right\|_{L_t^q L_x^p(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^{q'}L_x^{p'}(\mathbb{R}\times\mathbb{R}^d)} \tag{4}$$

holds true for

$$\frac{2}{q}+\frac{d}{p}=\frac{d}{2} \quad \textit{and} \quad \begin{cases} p\in[2,\infty] & \textit{if } d=1 \\ p\in[2,\infty) & \textit{if } d=2 \\ p\in\left[2,\frac{2d}{d-2}\right) & \textit{if } d\geq 3 \end{cases}$$

Remark 3. The relation between q, p and d can be obtained by scaling (3).
In $d = 2$ the endpoint $(q,p) = (2,\infty)$ has been proved false by Montgomery-Smith [MS97] with a counterexample involving Brownian motion.
For $d \geq 3$, the endpoint $(q,p) = \left(2, \frac{2d}{d-2}\right)$ has been proved by Keel and Tao [KT98].



Remark 4. The bound (4) is closely related to the bound for solution of the inhomogeneous Schrödinger equation:

$$\begin{cases} i\partial_t u - \triangle u = F \\ u(0,x) = u_0(x) \end{cases}$$

which by Duhamel's formula is

$$u(t,x) = e^{-it\triangle}u_0 + i\int_0^t e^{-i(t-s)\triangle}F(s)\,\mathrm{d}s. \tag{5}$$

We start proving L^p-bounds for the kernel in (4):

Lemma 1. *We have the following estimates:*

$$\left\|e^{-it\triangle}v\right\|_{L^2} = \|v\|_{L^2} \qquad\qquad \left\|e^{-it\triangle}v\right\|_{L^\infty} \leq (4\pi|t|)^{-\frac{d}{2}}\|v\|_{L^1}.$$

Energy estimate *Decay estimate*

Interpolating between them for $2 \leq p \leq \infty$ we obtain:

$$\left\|e^{-it\triangle}v\right\|_{L^p} \leq (4\pi|t|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|v\|_{L^{p'}}.$$

Proof of Theorem 3. From Lemma 1 applied to (4) we have:

$$\left\| \int_{-\infty}^{\infty} e^{-i(t-s)\triangle} F(s)\,\mathrm{d}s \right\|_{L^p_x} \leq \int_{-\infty}^{\infty} (4\pi|t-s|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|F(s)\|_{L^{p'}_x}\,\mathrm{d}s.$$

The RHS can be expressed as a convolution: call $f(t) = \|F(t)\|_{L^{p'}_x}$ and $g(t) = (4\pi|t|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}$, then

$$\|\mathrm{LHS}\|_{L^q_t(\mathbb{R})} \lesssim \|f * g\|_{L^q(\mathbb{R})}\,.$$

Using *weak Young inequality* for $r > 1$:

$$\|f * g\|_{L^q} \leq \|f\|_s \|g\|_{r,\infty} \qquad \text{for all } (s,r) : \frac{1}{s} + \frac{1}{r} = 1 + \frac{1}{q}.$$

In our case $g \in L^{r,\infty}(\mathbb{R})$ where $\frac{1}{r} = d\left(\frac{1}{2} - \frac{1}{p}\right)$. Notice that, by scaling, $\frac{1}{q} = \frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$, then $\frac{2}{q} = \frac{1}{r}$, which implies $s = q'$, and

$$\|\mathrm{LHS}\|_{L^q_t(\mathbb{R})} \lesssim \|f * g\|_{L^q(\mathbb{R})} \lesssim \|f\|_{q'} \|g\|_{r,\infty} = \|F\|_{L^{q'}_t L^{p'}_x(\mathbb{R}\times\mathbb{R}^d)}\,.$$

This proves the estimate apart from the endpoint. □

3 Endpoint Strichartz Estimates

To obtain the endpoint $(q,p) = \left(2, \frac{2d}{d-2}\right)$ in dimension $d \geq 3$ we rewrite the estimates (4) using the bilinear form:

$$T(F,G) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\langle (e^{-is\triangle})^\star F(s)\,,\, (e^{-it\triangle})^\star G(t) \right\rangle \mathrm{d}s\,\mathrm{d}t$$

where $\langle \cdot\,,\,\cdot\rangle$ is the $L^2(\mathbb{R}^d)$ scalar product. In this point the estimate (4) is equivalent to

$$|T(F,G)| \lesssim \|F\|_{L^2_t L^{p'}_x} \|G\|_{L^2_t L^{p'}_x}\,. \tag{6}$$

3.1 Dyadic decomposition of the Bilinear Estimate

We decompose our bilinear form T dyadically as

$$T(F,G) = \sum_{j\in\mathbb{Z}} T_j(F,G) \quad \text{where} \tag{7}$$

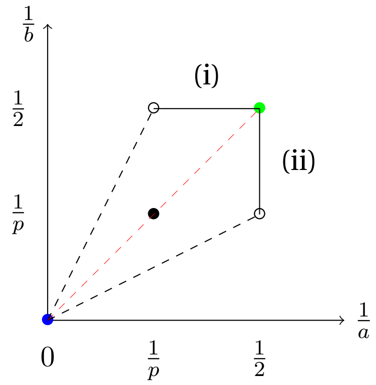
$$T_j(F,G) = \iint_{\{(t,s)\,:\, t-2^{j+1} < s \leq t-2^j\}} \left\langle (e^{-is\triangle})^\star F(s)\,,\, (e^{-it\triangle})^\star G(t) \right\rangle \mathrm{d}s\,\mathrm{d}t.$$

Idea of the proof: We start by showing the bound (6) for T_0. Let us interpolate

$$|T_0(F,G)| \lesssim \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t L^{b'}_x} \tag{8}$$

for $\bullet\ a=b=\infty$ $\bullet\ a=b=2$
By scaling this also gives the bound

$$|T_j(F,G)| \lesssim \|F\|_{L_t^2 L_x^{p'}} \|G\|_{L_t^2 L_x^{p'}} \qquad \text{for all } j \in \mathbb{Z}.$$



3.2 Better control on dyadic estimates

To bound the dyadic sum in (7) we need additional decay:

$$|T_j(F,G)| \lesssim 2^{-j\beta(a,b)} \|F\|_{L_t^2 L_x^{a'}} \|G\|_{L_t^2 L_x^{b'}} \tag{9}$$

for (a,b) in an open neighborhood of (p,p) and some

$$\beta(a,b) = \frac{d-2}{2} - \frac{d}{2}\left(\frac{1}{a} + \frac{1}{b}\right) \geq 0.$$

By scaling and interpolation this amounts to showing (8) for:

(i) $a=2, b \in (2,p)$,

(ii) $b=2, a \in (2,p)$.

Proof. By applying Cauchy-Schwarz and (4) (non-endpoint Strichartz) we get the point $a=b=2$. Time locality of T_0 and Hölder gives us the other estimates. □

3.3 Summing up the dyadic pieces in (7)

Assume that F and G have the form

$$F(t,x) = 2^{-k/p'} f(t) \mathbb{1}_{E(t)}(x), \qquad G(t,x) = 2^{-\tilde{k}/p'} g(t) \mathbb{1}_{\widetilde{E}(t)}(x)$$
$$|E(t)| \lesssim 2^k, \quad |\widetilde{E}(t)| \lesssim 2^{\tilde{k}} \qquad \forall t \in \mathbb{R}.$$

Then (9) simplifies to

$$|T_j(F,G)| \lesssim 2^{(k-j\frac{d}{2})(\frac{1}{p}-\frac{1}{a})+(\tilde{k}-j\frac{d}{2})(\frac{1}{p}-\frac{1}{b})} \|f\|_{L^2} \|g\|_{L^2}.$$

By choosing suitable (a,b) for any $(k,\tilde{k})$ we have

$$|T_j(F,G)| \lesssim 2^{-\epsilon(|k-j\frac{d}{2}|)+(|\tilde{k}-j\frac{d}{2}|)} \|f\|_{L^2} \|g\|_{L^2}$$

which is summable in $j \in \mathbb{Z}$.

Lemma 2 (Atomic decomposition of L^p)**.** *Let* $1<p<\infty$. *The* $F(t,\cdot) \in L_x^p$ *can be written as*

$$F(t,\cdot) = \sum_{k=-\infty}^{\infty} f_k(t) 2^{-k/p} \chi_{E_k(t)}(\cdot)$$

where $|\chi_{E_k(t)}| < \mathbb{1}_{E_k(t)}$ *with* $|E_k(t)| < 2^k$ *and*

$$\|f_k(t)\|_{\ell^p} \lesssim \|F(t,\cdot)\|_{L_x^p}.$$

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