Sharp Fourier restriction theory and Strichartz estimates

Gianmarco Brocchi

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Advisor: Dr. Diogo Oliveira e Silva

Second Advisor: Prof. Dr. Christoph Thiele

Mathematical Institute

Mathematisch-Naturwissenschaftliche Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

Abstract

This thesis studies extremizers for the Strichartz inequality for a family of fourth order Schrödinger equations on $\mathbb{R}$:

$$i\partial_t u - \mu\triangle u + \triangle^2 u = 0, \quad \mu \geqslant 0. \tag{1}$$

We solve an open problem that was posed in the literature about seven years ago by Jiang, Pausader and Shao in [JPS10] about the existence of maximizers for the corresponding Strichartz inequality when $\mu = 0$:

$$\|6^{\frac{1}{6}}|\nabla|^{\frac{1}{3}} e^{it\triangle^2} f\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})} \leqslant \mathbf{S}\,\|f\|_{L^2(\mathbb{R})}\,. \tag{2}$$

We prove that extremizers for (2) exist and are smooth.
To show this result we will use an orthogonal basis of polynomials in $L^2[-1,1]$. Connections between other Strichartz estimates and orthogonal polynomials have been recently discussed in [Gon17].

For proving smoothness of maximizers, we exploit the bootstrapping argument from [EHL11] and adapted in [HS12] for showing regularity for extremizers of the Airy–Strichartz inequality.

The study of extremizers is closely related to the research of optimal constants in the corresponding inequalities. For the Schrödinger equation:

$$i\partial_t u - \triangle u = 0 \tag{3}$$

Strichartz inequalities for the solution have been intensely studied in the last decade by Kunze [Kun03], Foschi [Fos07], Shao [Sha09], Oliveira e Silva and Quilodrán [OeSQ16]. For a more complete list, the reader can refer to [FOeS17]. In low dimensions the sharp constants have been calculated and extremizers characterised using different tools from PDEs and Harmonic Analysis, in particular from the theory of restriction for the Fourier transform.

The restriction problem was first posed by Stein: he wondered when it is possible to meaningfully restrict the Fourier transform on a subset E of the Euclidean space. Surprisingly, the answer is closely related to the *curvature* of the set E. In fact, curvature is one of the main factors to cause decay of oscillatory integrals.

Structure of this thesis

In the first chapter we introduce the Schrödinger equations together with some background: Oscillatory integrals, Restriction theory and Strichartz estimates.

In Chapter 2 we present the family of fourth order Schrödinger equation in (1). We focus on the case $\mu = 0$ and the corresponding Strichartz estimate (2).

Acknowledgement

I want to thank my advisor Diogo Oliveira e Silva for inspiring and supporting me along this path. I think he taught me much more than just mathematics.

I would also say a big "Thank you" to all friends I have found abroad.
They made me feel at home at every latitude.

Thanks to my family for being the latitude to whom I belong.

CONTENTS

1 PRELIMINARIES AND BACKGROUND

THE SCHRÖDINGER EQUATION

The Schrödinger equation is a differential equation that describes the evolution of a quantum system. It was introduced by Erwin Schrödinger in the 1925, who won the Nobel Prize for Physics eight years later. The simplified form the of equation we use in this thesis is

$$i\frac{\partial}{\partial t}\Psi(x,t) = [\triangle + V(x,t)]\,\Psi(x,t)$$

where i is the imaginary unit, Ψ is the so-called wave function and V is a potential.

When the potential V vanishes identically, we have the initial value problem for the homogeneous Schrödinger equation in $\mathbb{R}^d$ with initial datum u_0.

$$\begin{cases} i\partial_t u - \triangle u = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{SE}$$

Solution via Fourier analysis

When the initial datum u_0 is taken in the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ it is possible to give an explicit formula for the solution of the initial value problem (SE), by using the Fourier transform on $\mathbb{R}^d$.

Fourier transform	Inverse Fourier transform
$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx,$	$\check{f}(y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{iy\cdot\xi} f(\xi)\,d\xi.$

We will use the notation $đ\xi$ to indicate the normalised measure $\frac{d\xi}{(2\pi)^d}$. With this normalisation the Fourier transform is an isometry between $L^2(\mathbb{R}^d, dx)$ and $L^2\left(\mathbb{R}^d, đ\xi\right)$. More results about the Fourier transform are recalled in the Appendix A.
We denote with $\mathcal{F}_x$ the Fourier transform with respect to the space variable x, while $\mathcal{F}$ or $\mathcal{F}_{t,x}$ will stand for the space-time Fourier transform, in both time and space.
Applying the Fourier transform to (SE), we get

$$\begin{cases} i\partial_t \widehat{u} + |\xi|^2\widehat{u} = 0 \\ \widehat{u}(0,x) = \widehat{u_0}(x)\,, \quad \widehat{u_0} \in \mathscr{S}(\mathbb{R}^d). \end{cases}$$

The first line is now an algebraic equation. Divide it by the imaginary unit: since $i^{-1} = -i$ we obtain

$$\partial_t \widehat{u} - i|\xi|^2 \widehat{u} = 0.$$

Multiplying by $e^{-it|\xi|^2}$, we can rewrite the equation as

$$\partial_t (e^{-it|\xi|^2} \widehat{u}(t,\xi)) = 0.$$

Now integrate from 0 to t. Since $\lim_{t\to 0} e^{-it|\xi|^2} \widehat{u}(t,\xi) = \widehat{u}(0,\xi)$, we have

$$\int_0^t \partial_\tau (e^{-i\tau|\xi|^2} \widehat{u}(\tau,\xi))\, d\tau = e^{-it|\xi|^2} \widehat{u}(t,\xi) - \widehat{u_0}(\xi) = 0,$$

then

$$\widehat{u}(t,\xi) = e^{it|\xi|^2}\, \widehat{u}(0,\xi).$$

The right hand side is still a Schwartz function, so we are allowed to take the Inverse Fourier transform in the space variable on both sides:

$$u(t,x) = \mathcal{F}_x^{-1}\left(e^{it|\xi|^2}\, \widehat{u}_0(\xi)\right)(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{it|\xi|^2}\, \widehat{u}_0(\xi)\, d\!\!{}^-\xi.$$

Thus, our solution to (SE) is given by

$$u(t,x) = e^{-it\Delta} u_0, \tag{1.1.1}$$

where we indicated with $e^{-it\Delta}$ the evolution operator

$$(e^{-it\Delta} f)(t,x) := \mathcal{F}_x^{-1}(e^{it|\xi|^2} \widehat{f}(\xi)) = \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|^2)}\, \widehat{f}(\xi)\, d\!\!{}^-\xi.$$

The solution (1.1.1) enjoys the following symmetries.

- space-time translations: $u(t,x) \rightsquigarrow u(t+t_0, x+x_0)$, with $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$;
- parabolic dilations: $u(t,x) \rightsquigarrow u(\lambda^2 t, \lambda x)$, with $\lambda > 0$;
- change of scale: $u(t,x) \rightsquigarrow \mu u(t,x)$, with $\mu > 0$;
- space rotations: $u(t,x) \rightsquigarrow u(t, Rx)$, with $R \in SO(n)$;
- phase shifts: $u(t,x) \rightsquigarrow e^{i\theta} u(t,x)$, with $\theta \in \mathbb{R}$;
- Galilean transformations:

  $$u(t,x) \rightsquigarrow \exp\left(\frac{i}{4}\left(|v|^2 t + 2v\cdot x\right)\right) u(t, x+tv),$$

  with $v \in \mathbb{R}^n$.

Let G be the group generated by the above symmetries, and let g an element of G. If u solves (SE) with initial data u_0, then $v = g \cdot u$ is still a solution to (SE), where $\cdot$ denotes the multiplication on the group G.

Since we expressed the solution $u(t,x)$ to the Schrödinger equation (1.1.1) via the (space-time) Fourier transform, we can use results from the realm of oscillatory integrals theory in order to study regularity of the solution.

OSCILLATORY INTEGRALS

Oscillatory integrals are one of the main tools in harmonic analysis since its very beginning. The Fourier transform is in fact an example of oscillatory integral. We recall a few important results about them.

Notation 1.2.1. If x, y are real numbers, we write $x = \mathcal{O}(y)$ or $x \lesssim y$ if there exists a finite positive constant C such that $|x| \leqslant C|y|$. We write $x \sim y$ if $C^{-1}|y| \leqslant |x| \leqslant C|y|$ for some $C \neq 0$.

The main contribution in the oscillatory integral comes from critical points of the phase: those points in which the gradient of phase $\nabla\varphi$ vanishes. The following result is from [SS11b, Prop 2.1, Chapter 8, page 325].

Proposition 1.2.2 (Principle of non-stationary phase). *Let* $\varphi \in C^\infty(\mathbb{R}^d), \psi \in C_c^\infty(\mathbb{R}^d)$, *with* $|\nabla\varphi(x)| \geqslant c > 0$ *for every* $x \in \operatorname{supp}(\psi)$. *Then for any* $N \geqslant 0$

$$|I(\lambda)| = \left| \int_{\mathbb{R}^d} e^{i\lambda\varphi(x)}\psi(x)\,dx \right| \leqslant c_N \lambda^{-N} \qquad \forall \lambda > 0,$$

where the constant c_N *depends also on* φ *and* ψ.

When critical points are present, we cannot hope for such a decay.

Example 1. Consider the 1-dimensional case: let $a < 0 < b$ and let ψ be a smooth cut-off function supported in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and focus on the real part:

$$\Re\left(\int_a^b e^{i\lambda\varphi(x)}\psi(x)\,dx \right) = \int_a^b \cos(\lambda\varphi(x))\psi(x)\,dx\,.$$

As λ gets larger, we get more cancellation when no critical points are present. Below we fix $\lambda = 100$ and we plot two example: in the first one on the left $\varphi(x) = x$, who does not have critical points; in the second $\varphi(x) = x^2$, and the derivative vanishes at the origin.

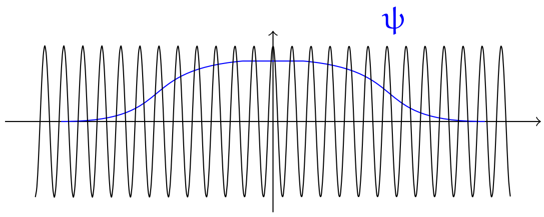


Figure 1.: Plot of the function $\cos(100x)$.

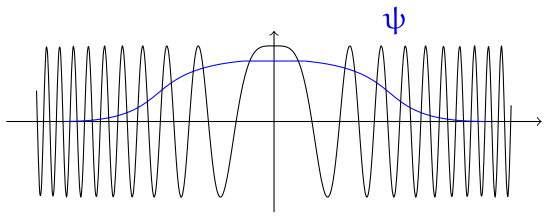


Figure 2.: Plot of the function $\cos(100x^2)$.

But we can still get some integrability if the critical points are not "too critical".

Lemma 1.2.3 (van der Corput). *Let* $\varphi \in C^2[a, b]$ *and* $|\varphi''(x)| \geqslant 1$ *for all* $x \in [a, b]$. *Then*

$$|I(\lambda)| = \left| \int_a^b e^{i\lambda\varphi(x)}\,dx \right| \leqslant \frac{8}{\lambda^{\frac{1}{2}}} \qquad \forall \lambda > 0.$$

The reader can find a proof in [SS11b, Prop 2.3, Chapter 8, page 328]

It is possible to partially generalize such result in higher dimension. See [CW02] for more details on what follows.

Definition 1.2.4. Let $\varphi \in C^\infty(\mathbb{R}^d)$. A point $x_0 \in \mathbb{R}^d$ is a *critical point* of φ if $\nabla\varphi(x_0) = 0$. The point x_0 is *non-degenerate* if the Hessian is non-degenerate in x_0, namely if

$$\nabla^2\varphi(x_0) = \left(\frac{\partial^2\varphi}{\partial x_i \partial x_j}(x_0)\right)_{i,j=1}^{d} \quad \text{has full rank}$$

or, equivalently, if $\det(\nabla^2\varphi(x_0)) \neq 0$.

Theorem 1.2.5 (van der Corput in higher dimension [CW02])**.** *Let* $\varphi \in C^\infty(\mathbb{R}^d), \psi \in C_c^\infty(\mathbb{R}^d)$*, with* $\det(\nabla^2\varphi(x)) \neq 0$ *for every* $x \in \operatorname{supp}(\psi)$*. Then*

$$I(\lambda) = \mathcal{O}(\lambda^{-\frac{d}{2}}).$$

These results have important applications in the theory of restriction of the Fourier transform.

RESTRICTION THEORY

Let f be an integrable function. Then its Fourier transform $\widehat{f}$ is a continuous function. Thus, for every subset $E \subset \mathbb{R}^d$, the restriction of $\widehat{f}$ to E makes sense as a continuous function. We can define a restriction operator $\mathcal{R}_E$ which maps into the space $C(E)$ of continuous functions on E.

$$\begin{aligned} \mathcal{R}_E \colon L^1(\mathbb{R}^d) &\to C(E) \\ f &\mapsto \widehat{f} \restriction_E \ . \end{aligned}$$

The operator $\mathcal{R}_E$ is a linear and bounded since

$$\|\widehat{f} \restriction_E\|_\infty \leqslant \|\widehat{f}\|_\infty = \sup_{\xi \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx \right| \leqslant \int_{\mathbb{R}^d} |f(x)|\, dx = \|f\|_{L^1}.$$

On the other hand, the restriction of an L^2 function to a null set[1] makes no sense.

We wonder if it is possible to make sense to this operator on $L^p(\mathbb{R}^d)$, for $1 < p < 2$. In other words, we wonder for which (q, p) and $E \subset \mathbb{R}^d$ the operator

$$\begin{aligned} \mathcal{R}_E \colon L^p(\mathbb{R}^d) &\to L^q(E) \\ f &\mapsto \widehat{f} \restriction_E \end{aligned}$$

1 when E is a set of zero Lebesgue measure in $\mathbb{R}^d$

is bounded, even when the subset E has zero Lebesgue measure.
First, we start with defining a class of "nice" subsets on which we would like to restrict $\widehat{f}$. We indicate with $(x^1, \dots, x^d)$ the coordinates of a vector $x \in \mathbb{R}^d$, and with $(p^0, p^1, \dots, p^d)$ the coordinates of a point $p \in \mathbb{R}^{d+1}$.

Definition 1.3.1. We say that M is a local smooth *hypersurface* in $\mathbb{R}^{d+1}$ if it is locally a graph of a smooth map

$$\begin{aligned} \varphi \colon \mathbb{R}^d &\to \mathbb{R} \\ x &\mapsto \varphi(x^1, \dots, x^d). \end{aligned}$$

This means that for every point $p \in M$ there exists a neighbourhood M_p of p and a map $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$M_p = \left\{ (x^0, x) \in \mathbb{R} \times \mathbb{R}^d \,:\, x^0 = \varphi(x^1, \dots, x^d) \right\}.$$

Example 2. The d-dimensional sphere $\mathbb{S}^d = \left\{ x \in \mathbb{R}^{d+1} \,:\, |x|^2 = 1 \right\}$ and the paraboloid $P = \left\{ (x, y) \in \mathbb{R}^d \times \mathbb{R} \,:\, y = |x|^2 \right\}$ are hypersurfaces in $\mathbb{R}^{d+1}$.

Let M be a smooth hypersurface in $\mathbb{R}^{d+1}$, and let $p \in M$. Then there exist an open neighbourhood A of p, a point $x_0 \in \mathbb{R}^d$, an open neighbourhood U of x_0 and map $\phi \in C^\infty(U)$ such that

$$\phi(x_0) = p, \qquad A \cap M = \phi(U) = \left\{ (x, y) \in U \times \mathbb{R} \,:\, y = \varphi(x^1, \dots, x^d) \right\}.$$

The map $\phi(x) = (x, \varphi(x))$ (usually called "chart") maps $\phi \colon U \to \phi(U)$. We can carry over the Lebesgue measure $\mathcal{L}^d$ on $\mathbb{R}^d$ to M via ϕ. Let f be a smooth function supported in the compact region V on M, with $V = \phi(U)$. We can define the surface measure σ on the surface M via change of variables:

$$\begin{aligned} \int_V f(v)\, d\sigma(v) &:= \int_{\phi(U)} f(v)\, d\sigma(v) \\ &= \int_U (f \circ \phi)(x)\, |\mathrm{Jac}(\phi(x))|\, dx = \int_U (f \circ \phi)(x) \sqrt{1 + |\nabla \varphi(x)|^2}\, dx. \end{aligned}$$

Up to translations and rotations, we can assume that $p = (x^0, x) = 0 \in \mathbb{R}^{d+1}$, $\phi(0) = (0, \varphi(0)) = 0$ and $\nabla \varphi(0) = 0$. Once a basis of $\mathbb{R}^d$ is fixed, the Hessian of φ at the origin can be represented as a matrix $\nabla^2 \varphi(0)$. This is a linear map on $\mathbb{R}^d$. Since φ is smooth, its Hessian is symmetric, so it has d real eigenvalues $k_1, \dots, k_d$.

Definition 1.3.2 (Curvature)**.** Let $p \in M$ and $x \in \mathbb{R}^d$, with $\phi(x) = (x, \varphi(x)) = p$ as before. The eigenvalues $k_1, \dots, k_d$ of the Hessian $\nabla^2 \varphi(x)$ are called *principal curvatures* of the M in p. The product of the eigenvalues

$$k(p) := k_1 \cdot \ldots \cdot k_d = \det(\nabla^2 \varphi(x))$$

is the *Gaussian curvature* of M at p.

Let μ be a smooth compactly supported measure on the hypersurface M. For example, consider $\mu = \psi\sigma$, where σ is the surface measure and $\psi \in C_c^\infty(M)$, where $C_c^\infty(M)$ is the space of smooth, compactly supported functions on the hypersurface M. Consider the Fourier transform of such a measure[2].

When M has non vanishing Gaussian curvature at every point, we can expect the Fourier transform of the measure μ to decay. In fact, we assume that $\text{supp}(\mu) = \phi(U)$ with U compact, then

$$\widehat{\mu}(\xi) = \int_{\phi(U)} e^{-i\xi\cdot y}\, d\mu(y) = \int_U e^{-i\xi\cdot\phi(x)} |\text{Jac}(\phi(x))|\, dx = \int_U e^{-i\xi\cdot\phi(x)} \left(1 + |\nabla\varphi(x)|^2\right)^{\frac{1}{2}} dx.$$

Expanding $\varphi(x)$ in the origin using Taylor, the first two terms are zero, so we have

$$\varphi(x) = \frac{1}{2}\sum_{j=1}^{d} k_j x_j^2 + \mathcal{O}(|x|^3).$$

When the principal curvatures k_j never vanish, we can apply Theorem 1.2.5 to get the decay

$$\widehat{\mu}(\xi) = \mathcal{O}(|\xi|^{-\frac{d}{2}}).$$

We want to see if the same result holds true when we consider functions on M. Let us introduce an operator to deal with this case.

Definition 1.3.3. Let $M \subset \mathbb{R}^{d+1}$ be a d-dimensional manifold and μ a smooth measure supported on it. We define the following operators

$$\begin{aligned} &\textit{Restriction operator} && \textit{Extension operator} \\ &\mathcal{R}\colon L^p(\mathbb{R}^{d+1}) \to L^2(M,\mu) && \mathcal{R}^\star\colon L^2(M,\mu) \to L^{p'}(\mathbb{R}^{d+1}) \\ &\qquad f \mapsto (\mathcal{F}f)\restriction_M && \qquad g \mapsto \mathcal{F}^{-1}(g\,\mu) \end{aligned} \tag{1.3.1}$$

The operator $\mathcal{R}^\star$ is the adjoint of $\mathcal{R}$. When the manifold M is curved and the measure μ is compactly supported (for example, consider the d-dimensional sphere $\mathbb{S}^d$ with it surface measure σ), in the case $q = 2$ one can use the following result due to Tomas and Stein [Tom75].

Theorem 1.3.4 (Tomas-Stein). *Let $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, and $f \in L^p(\mathbb{R}^{d+1})$, then*

$$\|\mathcal{R}f\|_{L^2(\mathbb{S}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^{d+1})} \quad \textit{holds for} \quad 1 \leqslant p \leqslant \frac{2(d+2)}{d+4}.$$

The dual statement for the extension operator reads:

2 A rigorous definition of Fourier transform of a measure is reported in Appendix A.

Theorem 1.3.5 (Dual Tomas-Stein). *Let* $\mathbb{S}^d \subset \mathbb{R}^{d+1}$*, and* $g \in L^2(\mathbb{S}^d)$*, then*

$$\|\mathcal{R}^\star g\|_{L^{p'}(\mathbb{R}^{d+1})} \lesssim \|g\|_{L^2(\mathbb{S}^d)} \quad \textit{holds for} \quad p' \geqslant 2 + \frac{4}{d}. \tag{1.3.2}$$

Remark 1.3.6. The implicit constants in Theorem 1.3.4 and Theorem 1.3.5 do not depend on the function f, nor on g.

BACK TO THE SCHRÖDINGER EQUATION

Look closer at the solution of (1.1.1):

$$u(t,x) = e^{-it\Delta}u_0 = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|^2)}\widehat{u_0}(\xi)\, d\xi.$$

We multiply and divide by 2π and we interpret the above formula as an inverse space-time Fourier Transform on $\mathbb{R}^{d+1}$:

$$u(t,x) = \int_{\mathbb{R}^d} e^{i(x\cdot\xi+t|\xi|^2)}\widehat{u_0}(\xi)\, đ\xi = \int_{\mathbb{R}^{d+1}} e^{i(t,x)\cdot(\tau,\xi)}\, 2\pi\,\widehat{u_0}(\xi)\,\delta(\tau - |\xi|^2)\frac{d\tau}{2\pi}\, đ\xi$$

from which:

$$u = \mathcal{F}^{-1}\big(\underbrace{\widehat{u_0}(\xi)}_{f(\xi)}\ \underbrace{2\pi\,\delta(\tau-|\xi|^2)}_{\mu(\tau,\xi)}\big) = \mathcal{F}^{-1}(f\,\mu). \tag{1.4.1}$$

where $\delta(\tau - |\xi|^2)$ is a measure supported on the paraboloid $P = \{(\tau,\xi) \in \mathbb{R}^{d+1}, \tau = |\xi|^2\}$.

Thus, the solution of the Schrödinger equation (SE) is given by applying the extension operator $\mathcal{R}^\star$ to the function $\widehat{u_0}$ when M is the paraboloid P, and μ is the measure $2\pi\,\delta(\tau - |\xi|^2)$.
The operator $e^{-it\Delta}$ is, in fact, the composition of $\mathcal{R}^\star$ with the spatial Fourier transform on $\mathbb{R}^d$.

$$u_0 \xmapsto{\ e^{-it\Delta}\ } u(t,x)$$
$$u_0 \mapsto \widehat{u_0} \mapsto \mathcal{R}^\star\widehat{u_0} = u(t,x)$$

Remark 1.4.1. The Tomas-Stein inequality (1.3.2) holds on *compact* hypersurface. In the case of the paraboloid, we can remove this assumption via rescaling[3]. Consider $u_0 \in L^2(\mathbb{R}^d)$ such that

$$\operatorname{supp}(\widehat{u_0}) \subseteq \mathbf{B}_1^d = \{\xi \in \mathbb{R}^d : |\xi| \leqslant 1\}.$$

Rescaling u_0 with $\lambda > 0$, the Fourier transform changes with the dual scaling:

$$(u_0)_\lambda(x) = u_0(\lambda x) \quad \Rightarrow \quad \widehat{(u_0)_\lambda}(\xi) = \lambda^{-d}\widehat{u_0}(\xi/\lambda) = \widehat{u_0}^\lambda(\xi),$$

3 This is particular of the paraboloid, it is not possible to do it, for example, for the hyperboloid

then $\widehat{u_0}^{\lambda}$ is supported on $\mathbf{B}^d_\lambda = \{\xi \in \mathbb{R}^d : |\xi| \leqslant \lambda\}$. The rescaled extension inequality

$$\|\mathcal{R}^\star \widehat{u_0}^{\lambda}\|_{L^{p'}(\mathbb{R}^{d+1})} = \lambda^{-\frac{d+2}{p'}} \|\mathcal{R}^\star \widehat{u_0}\|_{L^{p'}(\mathbb{R}^{d+1})} \leqslant C\lambda^{-\frac{d}{2}} \|\widehat{u_0}\|_{L^2(M)} = \|\widehat{u_0}^{\lambda}\|_{L^2(M)}$$

holds with constant $C_\lambda = C\lambda^{-\frac{d}{2}+\frac{d+2}{p'}}$. In particular, for the value $p' = 2 + \frac{4}{d}$ we have $C_\lambda = C$ for every $\lambda > 0$. From Theorem 1.3.5, letting $\lambda \to \infty$ we obtain the bound for the whole paraboloid P. Since functions with compactly supported Fourier transform are dense in L^2, with a limiting argument we obtain the extension inequality for all initial data in L^2.

Once we interpret the solution of the Schrödinger equation $e^{-it\Delta}u_0$ as a Fourier extension from a paraboloid, we can apply Theorem 1.3.5, that holds, by the previous remark, for $p(d) = 2 + \frac{4}{d}$. Then we have

$$\|e^{-it\Delta}u_0\|_{L^{p(d)}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)}. \tag{1.4.2}$$

The exponent $p(d)$ is also called the *Strichartz exponent*.

STRICHARTZ ESTIMATES

The Strichartz estimates are a family of inequalities for dispersive equations in which the norm of the solution is taken in the mixed Lebesgue space $L^q_t(\mathbb{R})L^p_x(\mathbb{R}^d)$.

They were introduced by Robert Strichartz in his seminal paper [Str77] in which he pointed out the connection between restriction theory of the Fourier transform and the decay of the solution of wave and Schrödinger equations.

Strichartz estimates for Schrödinger equation

The estimate we obtained in (1.4.2) using restriction theory is quite remarkable, nevertheless Theorem 1.3.5 gives estimates only on isotropic Lebesgue space (on $L^q_t(\mathbb{R})L^p_x(\mathbb{R}^d)$ when $q = p$). On the other hand, the paraboloid is invariant under the anisotropic scaling

$$(x, t) \rightsquigarrow (\lambda x, \lambda^2 t).$$

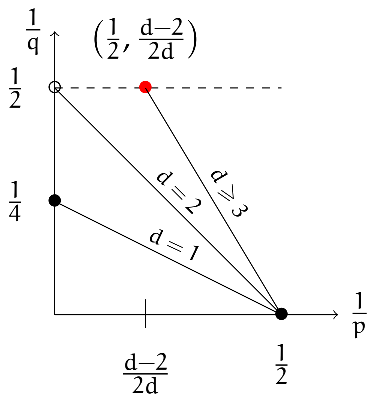
The solution of the Schrödinger equation (1.1.1), as a Fourier extension from a paraboloid, enjoys this not linear dilation in the time coordinate. So it is reasonable to study restriction and extension estimates on anisotropic spaces, i.e. when $q \neq p$. In fact, the solution of the Schrödinger equation (1.1.1) enjoys the following estimates:

$$\|e^{-it\Delta}u_0\|_{L^q_t(\mathbb{R})L^p_x(\mathbb{R}^d)} \leqslant C\|u_0\|_{L^2(\mathbb{R}^d)}. \tag{1.5.1}$$

The constant $C = C(q, p, d)$ depends on exponents and dimension. By scaling (1.5.1), we obtain the condition:

$$\frac{2}{q} + \frac{d}{p} = \frac{d}{2} \quad \text{with} \quad \begin{cases} p \in [2, \infty] & \text{if } d = 1 \\ p \in [2, \infty) & \text{if } d = 2 \\ p \in \left[2, \frac{2d}{d-2}\right] & \text{if } d \geqslant 3 \end{cases} \tag{1.5.2}$$

For a given dimension d, a pair (q, p) satisfying the above relation is called *admissible*.



Remark 1.5.1. In $d = 2$ the endpoint $(q, p) = (2, \infty)$ has been proved false by Montgomery-Smith [MS98]. For $d \geqslant 3$, the endpoint $(q, p) = \left(2, \frac{2d}{d-2}\right)$ has been proved by Keel and Tao [KT98].

Proving this inequality is equivalent to showing either of the following:

- $T := e^{-it\Delta} \quad : L^2(\mathbb{R}^d) \longrightarrow L^q_t L^p_x(\mathbb{R} \times \mathbb{R}^d)$ is bounded,
- $T^\star := (e^{-it\Delta})^\star \quad : L^{q'}_t L^{p'}_x(\mathbb{R} \times \mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is bounded.

The composition $TT^\star$:

- $TT^\star = e^{-it\Delta}(e^{-is\Delta})^\star \colon L_t^{q'}L_x^{p'}(\mathbb{R}\times\mathbb{R}^d) \to L_t^q L_x^p(\mathbb{R}\times\mathbb{R}^d)$ is a bounded operator.

We prove the last bound for $TT^\star$ and, by Hölder and duality, the previous follow.

Theorem 1.5.2 (Nonendpoint Strichartz estimates). *The operator $TT^\star$ is given by $u \mapsto \int_{-\infty}^{+\infty} e^{-i(t-s)\Delta} u \, ds$ and the following inequality holds*

$$\left\| \int_{-\infty}^{\infty} e^{-i(t-s)\Delta} F(s)\, ds \right\|_{L_t^q L_x^p(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^{q'} L_x^{p'}(\mathbb{R}\times\mathbb{R}^d)} \tag{1.5.3}$$

for (d, q, p) satisfying the condition (1.5.2).

Remark 1.5.3. The bound (1.5.3) is closely related to the bound for the solution of the inhomogeneous Schrödinger equation:

$$\begin{cases} i\partial_t u - \Delta u = F \\ u(0,x) = u_0(x) \end{cases}$$

which by Duhamel's formula is

$$u(t,x) = e^{-it\Delta}u_0 + i\int_0^t e^{-i(t-s)\Delta} F(s)\, ds. \tag{1.5.4}$$

We start proving L^p-bounds for the kernel in (1.5.3). Details of the proof can be found in [LP14].

Lemma 1.5.4. *The operator $e^{it\Delta}$ defined on $\mathscr{S}(\mathbb{R}^d)$ extends to an unitary operator on $L^2(\mathbb{R}^d)$. Moreover, we have the following estimates:*

$$\underbrace{\|e^{-it\Delta}v\|_{L^2} = \|v\|_{L^2}}_{\textit{Energy estimate}} \qquad \underbrace{\|e^{-it\Delta}v\|_{L^\infty} \leqslant (4\pi|t|)^{-\frac{d}{2}} \|v\|_{L^1}.}_{\textit{Decay estimate}}$$

Interpolating between them for $2 \leqslant p \leqslant \infty$ we obtain:

$$\|e^{-it\Delta}v\|_{L^p} \leqslant (4\pi|t|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|v\|_{L^{p'}}.$$

Proof of Theorem 1.5.2. We prove the theorem in all case but the endpoint.
From Lemma 1.5.4 applied to (1.5.3) we have:

$$\left\| \int_{-\infty}^{\infty} e^{-i(t-s)\Delta} F(s)\, ds \right\|_{L_x^p} \leqslant \int_{-\infty}^{\infty} (4\pi|t-s|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|F(s)\|_{L_x^{p'}}\, ds.$$

The right hand side can be expressed as a convolution. Let A be the left hand side and let $f(t) = \|F(t)\|_{L_x^{p'}}$ and $g(t) = (4\pi|t|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}$, then

$$\|A\|_{L_t^q(\mathbb{R})} \lesssim \|f * g\|_{L_t^q(\mathbb{R})}.$$

Using *weak Young inequality*[4] for $r > 1$:

$$\|f * g\|_{L^q} \leqslant \|f\|_s \|g\|_{r,\infty} \quad \text{for all } (s,r) : \frac{1}{s} + \frac{1}{r} = 1 + \frac{1}{q}.$$

In our case $g \in L^{r,\infty}(\mathbb{R})$ where $\frac{1}{r} = d\left(\frac{1}{2} - \frac{1}{p}\right)$. Notice that, by scaling, we have

$$\frac{1}{q} = \frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right) \quad \text{then} \quad \frac{2}{q} = \frac{1}{r},$$

which implies $s = q'$. Thus

$$\|A\|_{L^q_t(\mathbb{R})} \lesssim \|f * g\|_{L^q_t(\mathbb{R})} \lesssim \|f\|_{q'} \|g\|_{r,\infty} = \|F\|_{L^{q'}_t L^{p'}_x(\mathbb{R}\times\mathbb{R}^d)}.$$

This proves the estimate away from the endpoint. □

EXTREMIZERS FOR STRICHARTZ ESTIMATES

We are interested in the best value of the constant in the inequality (1.5.1). This will be defined as

$$\mathbf{C} := \sup_{u_0 \in L^2(\mathbb{R}^d)\setminus\{0\}} \frac{\|e^{-it\Delta}u_0\|_{L^q_t(\mathbb{R})L^p_x(\mathbb{R}^d)}}{\|u_0\|_{L^2(\mathbb{R}^d)}}. \tag{1.6.1}$$

Definition 1.6.1. A nonzero function f that realises equality in an inequality is called *extremizer* or *maximizer* for that inequality.

In particular, we look for a nonzero $f \in L^2(\mathbb{R}^d)$ that realises equality in (1.5.1), if there exist one. Even if maximizers do not exist, we always have sequences of functions maximising (1.6.1).

Definition 1.6.2. A sequence $\{f_n\}_{n\in\mathbb{N}}$, with $\|f_n\|_{L^2} \leqslant 1$ is an *extremizing sequence* for (1.5.1) if

$$\lim_{n\to\infty} \|e^{-it\Delta}f_n\|_{L^q_t(\mathbb{R})L^p_x(\mathbb{R}^d)} \to \mathbf{C}.$$

Remark 1.6.3. Due to the several symmetries of the solution, extremizing sequences may not converge to an extremizer in the strong topology.

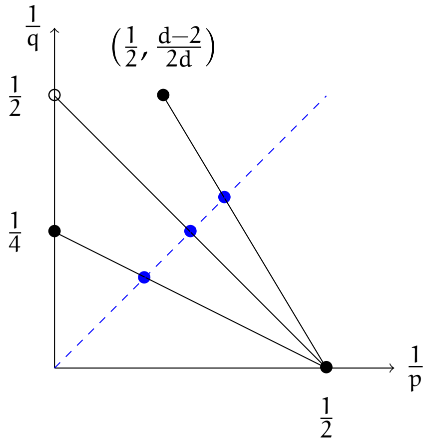
Despite of the difficulties, existence of extremizers for the Strichartz estimate (1.5.1) has been proved in *all* dimensions! The first result, in dimension 1, was given by Kunze [Kun03]: he proved existence of extremizers exploiting the concentration-compactness principle of Lions [Lio85]. Then Foschi [Fos07] managed to characterise maximizers in dimensions $d = 1$ and 2, showing that they are[5] Gaussians. A few

4 or by applying Hardy-Littlewood-Sobolev lemma
5 up to the symmetries of the solution Section 1.1.1.

years later Shao [Sha09] proved the existence of extremizers in every dimension, and not only in the symmetric case but for *any* non-endpoint admissible pair (q, p).

The figure shows admissible pairs of exponent (q, p) for different dimensions.
The blue dots on the diagonal represent the symmetric exponents (q, q), for which we can use restriction theory.

Characterise the extremizers in higher dimension is still an open problem.



2 A FAMILY OF FOURTH ORDER SCHRÖDINGER EQUATION

We study a family of fourth order Schrödinger equations in one dimension depending on the parameter $\mu \geqslant 0$[1]:

$$\begin{cases} i\partial_t u - \mu\triangle u + \triangle^2 u = 0 & x, t \in \mathbb{R} \\ u(0,x) = f(x) \in L^2(\mathbb{R}) \end{cases} \tag{2.0.1}$$

The solution of the equation (2.0.1) is given by

$$S_\mu(t)f := e^{it(\triangle^2 - \mu\triangle)}f = (e^{i(x\xi + t\phi_\mu(\xi))}\widehat{f}(\xi))^{\check{}} = \int_{\mathbb{R}} e^{i(x\xi+t\phi_\mu(\xi))}\widehat{f}(\xi)\, đ\xi, \tag{2.0.2}$$

where $\phi_\mu(\xi) = \xi^4 + \mu\xi^2$. A family of Strichartz estimates is available:

$$\|D_\mu^{\frac{\theta}{2}} S_\mu(t)f\|_{L_t^q(\mathbb{R})L_x^p(\mathbb{R})} \leqslant c\,\|f\|_{L^2(\mathbb{R})}, \tag{2.0.3}$$

where $(q,p) = (\frac{4}{\theta}, \frac{2}{1-\theta})$, $\theta \in [0,1]$, and the operator D_μ^α is given by

$$D_\mu^\alpha f(x) := \int_{\mathbb{R}} e^{ix\xi}|\phi''_\mu(\xi)|^{\frac{\alpha}{2}}\widehat{f}(\xi)\, d\xi. \tag{2.0.4}$$

These estimates have been proven by Kenig, Ponce and Vega [KPV91, Theorem 2.1] for a broad class of phase function ϕ. In their paper the authors deal with global and local smoothing properties of dispersive equations. These results are obtained exploiting the decay of the oscillatory integral representing the solution.

In our case, when the parameter θ ranges in $[0,1]$, we obtain all the points in line connecting $(\frac{1}{2}, 0)$ and $(0, \frac{1}{4})$ in the diagram. We are mainly interested in the symmetric case, when $q = p = 6$, and the inequality (2.0.3) is:

$$\|D_\mu^{\frac{1}{3}} S_\mu(t)f\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})} \leqslant C\,\|f\|_{L^2(\mathbb{R})}. \tag{2.0.5}$$

This case (in blue in the diagram) is obtained when $\theta = \frac{2}{3}$.

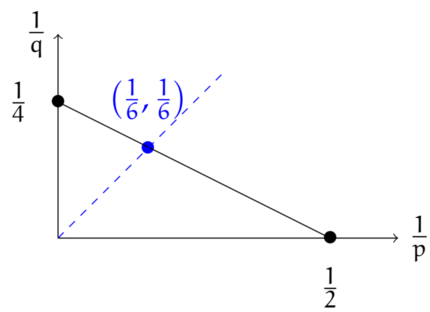


Figure 3.: Riesz diagram for the Strichartz estimates in (2.0.3).

1 The case $\mu < 0$ is not considered since Strichartz estimates may not be available in view of the presence of the critical point for the phase function, see [KPV91, Condition (2.1.c)].

Recently, in [OeSQ16] the authors studied the problem of existence of extremizers for the Strichartz estimates for the same family (2.0.1) but in dimension 2. By using a dichotomy result about existence of extremizers [JSS14, Section 4] for the corresponding Strichartz inequality they proved that maximizers exist when $\mu = 0$, and fail to exist when $\mu = 1$ (and, by scaling, when $\mu > 0$).

Aiming to a similar result for the 1-dimensional problem, in this thesis we study the problem of maximizers for the endpoint $\mu = 0$.

PURE POWER OF LAPLACIAN

From now on we will consider the case $\mu = 0$ for which our family (2.0.1) reduces to the equation:

$$\begin{cases} i\partial_t u + \triangle^2 u = 0 & x, t \in \mathbb{R} \\ u(0, x) = f(x) \in L^2(\mathbb{R}). \end{cases} \tag{2.1.1}$$

The solution is given by

$$S_0(t)f := e^{it\triangle^2} f = \left(e^{i(x\xi + t\xi^4)}\widehat{f}(\xi)\right)^{\vee} = \int_{\mathbb{R}} e^{i(x\xi+t\xi^4)}\widehat{f}(\xi)\, đ\xi\,. \tag{2.1.2}$$

As before, for the solution we have the corresponding Strichartz estimate:

$$\|D_0^{\frac{1}{3}} e^{it\triangle^2} f\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})} \leqslant \mathbf{S}\, \|f\|_{L^2(\mathbb{R})}\,, \tag{2.1.3}$$

where the operator $D_0^{\frac{1}{3}}$ is defined as

$$D_0^{\frac{1}{3}} f(x) := \int_{\mathbb{R}} e^{ix\xi} |6\xi^2|^{\frac{1}{6}} \widehat{f}(\xi)\, d\xi.$$

Remark 2.1.1. Here the operator $D_0^{\frac{1}{3}} = 6^{\frac{1}{6}} |\nabla|^{\frac{1}{3}}$. This operator differs from the one in (2.0.4) and in [KPV91] by a factor of $2^{\frac{1}{6}}$.

For the convenience of the reader, we indicate with $T(t)$ the propagator given by the composition $D_0^{\frac{1}{3}} S_0(t) = D_0^{\frac{1}{3}} e^{it\triangle^2}$. This is defined as

$$T(t)f(x) := \int_{\mathbb{R}} e^{ix\xi} 6^{\frac{1}{6}} \sqrt{w(\xi)} e^{it\xi^4} \widehat{f}(\xi)\, đ\xi, \qquad w(\xi) = |\xi|^{\frac{2}{3}}. \tag{2.1.4}$$

Sometimes we will omit the time variable t writing Tf in place of $T(t)f$.

The optimal constant $\mathbf{S}$ in (2.1.3) is defined as

$$\mathbf{S} := \sup_{\substack{f \in L^2(\mathbb{R}), \\ f \neq 0}} \frac{6^{\frac{1}{6}}\, \||\nabla|^{\frac{1}{3}} e^{it\triangle^2} f\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})}}{\|f\|_{L^2(\mathbb{R})}}. \tag{2.1.5}$$

We write this quantity in a different way. First, we expand the solution $T(t)f$ in (2.1.4) using the extension operator $\mathcal{R}^\star$ defined in (1.3.1). Let $\nu(\xi,\tau) = \delta\left(\tau - \xi^4\right)$. Then

$$T(t)f = D_0^{\frac{1}{3}} e^{it\triangle^2} f = \mathcal{F}^{-1}_{t,x}\left(2\pi \widehat{f}(\xi)\sqrt{w(\xi)}\nu(\xi,\tau)\right).$$

Since $\|\cdot\|^3_{L^6} = \|[\cdot]^3\|_{L^2}$, we get

$$\mathbf{S}^3 = \sup_{f\in L^2} \frac{6^{\frac{1}{2}}\,\| |\nabla|^{\frac{1}{3}} e^{it\triangle^2} f\|^3_{L^6_{t,x}(\mathbb{R}^2)}}{\|f\|^3_{L^2(\mathbb{R})}} = \sup_{f\in L^2} \frac{6^{\frac{1}{2}}\,\left\|\left[\mathcal{F}^{-1}(2\pi \widehat{f}(\xi)\sqrt{w(\xi)}\nu(\xi,\tau))\right]^3\right\|_{L^2_{t,x}(\mathbb{R}^2)}}{\|f\|^3_{L^2(\mathbb{R})}}.$$

We focus on the quantity inside the norm in the numerator. By applying Plancherel (A.0.1) in dimension 2 we have

$$\left\|\left[\mathcal{F}^{-1}(2\pi \widehat{f}(\xi)\sqrt{w(\xi)}\nu(\xi,\tau))\right]^3\right\|_{L^2_{t,x}(\mathbb{R}^2)} = (2\pi)^{-1}\left\|\mathcal{F}\left[\mathcal{F}^{-1}(2\pi \widehat{f}(\xi)\sqrt{w(\xi)}\nu(\xi,\tau))\right]^3\right\|_{L^2_{t,x}(\mathbb{R}^2)}$$

We use the convolution identity for the Fourier transform (A.0.2) twice and the Inversion formula (A.0.4) to obtain

$$\begin{aligned}(2\pi)^{-1}\left\|\mathcal{F}\left[\mathcal{F}^{-1}(2\pi \widehat{f}\sqrt{w}\nu\right]^3\right\|_{L^2_{t,x}(\mathbb{R}^2)} &= (2\pi)^{-1}\left\|(2\pi)^{-1}\mathcal{F}\left[\mathcal{F}^{-1}(2\pi \widehat{f}\sqrt{w}\nu)\right]^2 * \widehat{f}\sqrt{w}\nu\right\|_{L^2_{t,x}} \\ &= (2\pi)^{-1}\|(2\pi)^{-1}\widehat{f}\sqrt{w}\nu * (\widehat{f}\sqrt{w}\nu * \widehat{f}\sqrt{w}\nu)\|_{L^2_{t,x}} \\ &= (2\pi)^{-2}\|\widehat{f}\sqrt{w}\nu * (\widehat{f}\sqrt{w}\nu * \widehat{f}\sqrt{w}\nu)\|_{L^2_{t,x}}.\end{aligned}$$

We apply the Fourier transform to the denominator, and again by Plancherel we have

$$\|f\|^3_{L^2(\mathbb{R})} = (2\pi)^{-\frac{3}{2}}\|\widehat{f}\|^3_{L^2(\mathbb{R})}.$$

Finally, since the Fourier transform is a bijection on L^2:

$$\mathbf{S}^3 = \sqrt{\frac{6}{2\pi}}\sup_{f\in L^2}\frac{\|\widehat{f}\sqrt{w}\nu * \widehat{f}\sqrt{w}\nu * \widehat{f}\sqrt{w}\nu\|_{L^2_{t,x}(\mathbb{R}^2)}}{\|\widehat{f}\|^3_{L^2(\mathbb{R})}} = \sqrt{\frac{3}{\pi}}\sup_{f\in L^2}\frac{\|f\sqrt{w}\nu * f\sqrt{w}\nu * f\sqrt{w}\nu\|_{L^2_{t,x}(\mathbb{R}^2)}}{\|f\|^3_{L^2(\mathbb{R})}}.$$

Taking the square to both sides, we have:

$$\frac{\pi}{3}\mathbf{S}^6 = \sup_{\substack{f\in L^2(\mathbb{R}),\\ f\neq 0}}\frac{\|f\sqrt{w}\nu * f\sqrt{w}\nu * f\sqrt{w}\nu\|^2_{L^2_{t,x}(\mathbb{R}^2)}}{\|f\|^6_{L^2(\mathbb{R})}} \tag{2.1.6}$$

EXISTENCE OF MAXIMIZERS

In this section we prove the following theorem.

Theorem 2.2.1. *There exist a maximizer for the Strichartz inequality (2.1.3).*

Then, the best constant in (2.1.5) is given by

$$\mathbf{S} = \sup_{\substack{f\in L^2(\mathbb{R}),\\ f\neq 0}} \frac{6^{\frac{1}{6}}\, \||\nabla|^{\frac{1}{3}} e^{it\triangle^2} f\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})}}{\|f\|_{L^2(\mathbb{R})}} = \max_{\substack{f\in L^2(\mathbb{R}),\\ f\neq 0}} \frac{6^{\frac{1}{6}}\, \||\nabla|^{\frac{1}{3}} e^{it\triangle^2} f\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})}}{\|f\|_{L^2(\mathbb{R})}}.$$

In [JPS10, Theorem 1.8] the authors proved a dichotomy result for extremizers of our Strichartz estimate:

Theorem 2.2.2 (Dichotomy, [JPS10])**.** *Either*

(i) $\mathbf{S} = S_{\mathrm{Schr}}$ *and there exist* $f \in L^2(\mathbb{R})$ *and a sequence* $\{a_n\}_{n\in\mathbb{N}}$ *going to infinity as* $n \to \infty$*, such that* $\{e^{ixa_n} f\}_{n\in\mathbb{N}}$ *is an extremizing sequence for* (2.1.5), or

(ii) $\mathbf{S} \neq S_{\mathrm{Schr}}$ *and extremizers for* (2.1.5) *exist.*

A solution to the extremizing problem (2.1.5) is related to the one for the classical Schrödinger equation. We recall that the sharp constant for the Strichartz estimate for the free propagator $e^{-it\triangle}$ is

$$S_{\mathrm{Schr}} := \sup_{v\in L^2(\mathbb{R})\setminus\{0\}} \frac{\|e^{-it\triangle} v\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})}}{\|v\|_{L^2(\mathbb{R})}} = \left(\frac{1}{12}\right)^{\frac{1}{12}}. \tag{2.2.1}$$

This constant was calculated by Foschi, see [Fos07, Theorem 1.1].

To prove the existence of extremizers it is enough to show a lower bound for $\mathbf{S}$ good enough to ensure that

$$\mathbf{S} > S_{\mathrm{Schr}}. \tag{2.2.2}$$

This will rule out the first case in Theorem 2.2.2. The condition (2.2.2) is equivalent to

$$\frac{\pi}{3}\mathbf{S}^6 > \frac{\pi}{3}(S_{\mathrm{Schr}})^6 = \frac{\pi}{6\sqrt{3}},$$

and because of (2.1.6), it is also equivalent to

$$\sup_{\substack{f\in L^2(\mathbb{R}),\\ f\neq 0}} \frac{\|f\sqrt{w}\nu * f\sqrt{w}\nu * f\sqrt{w}\nu\|^2_{L^2(\mathbb{R}^2)}}{\|f\|^6_{L^2(\mathbb{R})}} > \frac{\pi}{6\sqrt{3}}. \tag{2.2.3}$$

We can approximate the left hand side with some explicit function $f \in L^2$.

We start proving a result analogous to the [OeSQ16, Lemma 6.1] adapted for the 3-fold convolution where the perturbation is $\Psi(x) = x^4$.

Lemma 2.2.3. *Consider the measure* $\nu(y,t) = \delta\big(t - y^4\big)\,dy\,dt$*. Let* E *denote the support of the convolution measure* $\nu * \nu * \nu$*. Let* $w(x)$ *be the non-negative function* $|x|^{\frac{2}{3}}$ *and consider* $f(x) = e^{-x^4}\sqrt{w(x)} \in L^2(\mathbb{R})$*, then the convolution* $f\sqrt{w}\,\nu * f\sqrt{w}\,\nu * f\sqrt{w}\,\nu \in L^2(\mathbb{R}^2)$ *and the following lower bound holds:*

$$\frac{\|f\sqrt{w}\,\nu * f\sqrt{w}\,\nu * f\sqrt{w}\,\nu\|^2_{L^2(\mathbb{R}^2)}}{\|f\|^6_{L^2(\mathbb{R})}} \geqslant \frac{\|f\|^6_{L^2(\mathbb{R})}}{\int_E e^{-2\tau}\,d\tau\,d\xi}. \tag{2.2.4}$$

Proof. The following identity holds:

$$f^2\nu * f^2\nu * f^2\nu(\xi,\tau) = e^{-\tau}\, f\sqrt{w}\nu * f\sqrt{w}\nu * f\sqrt{w}\nu(\xi,\tau). \tag{2.2.5}$$

Moreover we have

$$\int_{\mathbb{R}^2} f^2\nu * f^2\nu * f^2\nu(\xi,\tau)\,d\xi\,d\tau = \|f\|^6_{L^2(\mathbb{R})}. \tag{2.2.6}$$

Then using the preceding identities and Cauchy-Schwarz we obtain

$$\begin{aligned}\|f\|^6_{L^2(\mathbb{R})} &= \int_{\mathbb{R}^2} e^{-\tau}(f\sqrt{w}\nu * f\sqrt{w}\nu * f\sqrt{w}\nu)(\xi,\tau)\,d\tau\,d\xi \\ &\leqslant \left(\int_E e^{-2\tau}\,d\tau\,d\xi\right)^{\frac{1}{2}} \|f\sqrt{w}\nu * f\sqrt{w}\nu * f\sqrt{w}\nu\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

that implies the desired inequality (2.2.4). □

We can explicitly calculate the right hand side of (2.2.4). The function $f(x) = e^{-x^4}\sqrt{w(x)}$ is even, and we have

$$\|f\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} e^{-2x^4}(x^2)^{\frac{1}{3}}\,dx = \frac{1}{2}\int_0^\infty e^{-2z}z^{\frac{5}{12}-1}\,dz = \frac{1}{2}\frac{\Gamma(\frac{5}{12})}{2^{\frac{5}{12}}}, \qquad \text{so} \qquad \|f\|^6_{L^2} = \frac{1}{2^4}\frac{\Gamma(\frac{5}{12})^3}{2^{\frac{1}{4}}}.$$

We compute the denominator. The support of $\nu * \nu * \nu$ is $E = \left\{(\xi,\tau) \in \mathbb{R}^2 : \tau \geqslant \frac{\xi^4}{27}\right\}$, see Proposition 2.2.4.

$$\int_E e^{-2\tau}\,d\tau\,d\xi = \int_{\mathbb{R}}\int_{\frac{1}{27}}^\infty e^{-2\lambda\xi^4}\xi^4\,d\lambda\,d\xi = \frac{1}{2}\int_{\frac{1}{27}}^\infty \left(\int_0^\infty e^{-2u}u^{\frac{5}{4}-1}\,du\right)\frac{d\lambda}{\lambda^{\frac{1}{4}+1}} = \frac{3^{\frac{3}{4}}\,\Gamma(\frac{5}{4})}{2^{\frac{1}{4}}}.$$

Putting all together, we obtain the lower bound:

$$\frac{\|f\|^6_{L^2(\mathbb{R})}}{\int_E e^{-2\tau}\,d\tau\,d\xi} = \frac{1}{2^4 3^{\frac{3}{4}}}\frac{\Gamma(\frac{5}{12})^3}{\Gamma(\frac{5}{4})} \approx 0.2913141\ldots \tag{2.2.7}$$

Unfortunately, this quantity is not large enough to defeat the sharp constant, in fact:

$$\frac{\pi}{3}(S_{\mathrm{Schr}})^6 = \frac{\pi}{6\sqrt{3}} \approx 0.302299 > 0.2913141.$$

The inequality in Lemma 2.2.3 was too crude and we need a new way to approximate $\|f\sqrt{w}\nu * f\sqrt{w}\nu * f\sqrt{w}\nu\|^2_{L^2(\mathbb{R}^2)}$. For this purpose, we exploit the identities (2.2.9), (2.2.5), and the following properties of the convolution measure.

Proposition 2.2.4. *Let $w(\xi) = |\xi|^{\frac{2}{3}}$ and ν be the measure defined by*

$$\nu(\xi,\tau) = \delta\big(\tau - |\xi|^4\big)\, d\xi\, d\tau.$$

Then the following properties hold for the convolution measure $w\nu * w\nu * w\nu$.

(a) It is absolutely continuous with respect the Lebesgue measure on $\mathbb{R}^2$.

(b) Its support is given by

$$E = \{(\xi,\tau) \in \mathbb{R}^2 \,:\, \tau \geqslant 3^{-3}|\xi|^4\}.$$

(c) It is radial *and* homogeneous *of degree zero in ξ, the sense that:*

$$(w\nu * w\nu * w\nu)(\lambda\xi, \lambda^4\tau) = (w\nu * w\nu * w\nu)(\xi,\tau), \quad \textit{for every } \lambda > 0.$$

$$\textit{and} \quad (w\nu * w\nu * w\nu)(-\xi,\tau) = (w\nu * w\nu * w\nu)(\xi,\tau) \quad \textit{for every } \xi \in \mathbb{R}.$$

Following the work of Oliveira e Silva e Quilodrán in [OeSQ16, Proposition 6.4] we will prove a more general result. The proof of the previous Proposition will follow from this taking the power $p = 4$.

Proposition 2.2.5. *Given $p \geqslant 2$, let $w(\xi) = |\xi|^{\frac{p-2}{3}}$. Let ν_p be the measure defined by*

$$\nu_p(\xi,\tau) = \delta(\tau - |\xi|^p)\, d\xi\, dt.$$

The following assertions hold Then the following properties hold for the convolution measure $w\nu_p * w\nu_p * w\nu_p$.

(a) It is absolutely continuous with respect the Lebesgue measure on $\mathbb{R}^2$.

(b) Its support is given by

$$E_p = \{(\xi,\tau) \in \mathbb{R}^2 \,:\, \tau \geqslant 3^{1-p}|\xi|^p\}.$$

(c) It is radial *in ξ, and* homogeneous *of degree zero in ξ, in the sense that:*

$$(w\nu_p * w\nu_p * w\nu_p)(\lambda\xi, \lambda^p\tau) = (w\nu_p * w\nu_p * w\nu_p)(\xi,\tau), \quad \textit{for every } \lambda > 0,$$

$$\textit{and} \quad (w\nu_p * w\nu_p * w\nu_p)(-\xi,\tau) = (w\nu_p * w\nu_p * w\nu_p)(\xi,\tau) \quad \textit{for every } \xi \in \mathbb{R}.$$

Proof. With the change of variables: $\eta \mapsto \frac{2}{3}\xi + \eta$, $\zeta \mapsto \frac{\xi}{3} - \zeta$, we can write the convolution measure as

$$(w\nu_p * w\nu_p * w\nu_p)(\xi,\tau) = \iint_{\mathbb{R}_\eta \times \mathbb{R}_\zeta} A_{\xi,\tau}(\eta,\zeta)\, d\eta\, d\zeta,$$

where

$$A_{\xi,\tau}(\eta,\zeta) := \delta(\tau - |\tfrac{\xi}{3} - \eta|^p - |\tfrac{\xi}{3} - \zeta|^p - |\tfrac{\xi}{3} + \eta + \zeta|^p) \left(|\tfrac{\xi}{3} - \eta||\tfrac{\xi}{3} - \zeta||\tfrac{\xi}{3} + \eta + \zeta|\right)^{\frac{p-2}{3}}.$$

(a),(c) It is enough to change variables and use that $\delta(\lambda^p F(x)) = \lambda^{-p}\delta(F(x))$ (see [FOeS17, Appendix]). Note also that

$$\begin{aligned}(w\nu_p * w\nu_p * w\nu_p)(-\xi,\tau) &= \iint A_{-\xi,\tau}(\eta,\zeta)\, d\eta\, d\zeta \\ &= \iint A_{\xi,\tau}(-\eta,-\zeta)\, d\eta\, d\zeta = (w\nu_p * w\nu_p * w\nu_p)(\xi,\tau).\end{aligned}$$

(b) First we show that every point in E_p belongs to the support of $w\nu_p * w\nu_p * w\nu_p$. In fact, let $\psi(y) = |y|^p$ and consider $y_1, y_2, y_3 \in \mathbb{R}$ such that

$$\xi = y_1 + y_2 + y_3, \quad \tau = \psi(y_1) + \psi(y_2) + \psi(y_3).$$

From the midpoint convexity of ψ it follows:

$$\frac{1}{3}\tau = \frac{1}{3}\psi(y_1) + \frac{1}{3}\psi(y_2) + \frac{1}{3}\psi(y_3) \geqslant \psi\left(\frac{y_1+y_2+y_3}{3}\right) = \psi(\xi/3). \tag{2.2.8}$$

On the other hand, consider $(\xi,\tau) \in \mathbb{R}^2$ such that $\tau \geqslant 3\psi(\xi/3)$. We want to find $y_1, y_2, y_3 \in \mathbb{R}$ as before. It is enough to find y_1, y_2, since $y_3 = \xi - (y_1 + y_2)$. The left hand side of Eq. (2.2.8) is convex and continuous, and it goes to infinity as $|(y_1,y_2)| \to \infty$. Then, for every fixed $\tau \geqslant 3\psi(\xi/3)$, for the Intermediate Values Theorem, there exists $(y_1,y_2) \in \mathbb{R}^2$ such that

$$\tau = \psi(y_1) + \psi(y_2) + \psi(\xi - (y_1+y_2)) \geqslant 3\psi(\xi/3).$$

□

We have that

$$\|f\sqrt{w\nu} * f\sqrt{w\nu} * f\sqrt{w\nu}\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} e^{-2\tau}(w\nu * w\nu * w\nu)^2(\xi,\tau)\, d\xi\, d\tau,$$

because, in the support of the measure, we can write $\tau = \lambda\xi^4$, for $\lambda \geqslant 3^{-3}$, and the following equality holds

$$(f\sqrt{w\nu} * f\sqrt{w\nu} * f\sqrt{w\nu})(\xi,\tau) = e^{-\tau}\,(w\nu * w\nu * w\nu)(\xi,\tau). \tag{2.2.9}$$

Applying Fubini and the change of variables $\lambda\xi^4 = u$, we get

$$\int_{\mathbb{R}^2} e^{-2\tau}(w\nu * w\nu * w\nu)^2(\xi,\tau)\,d\xi\,d\tau = \int_{\frac{1}{27}}^{\infty} (w\nu * w\nu * w\nu)^2(1,\lambda)\,2\int_0^{\infty} e^{-u}\left(\frac{u}{\lambda}\right)^{\frac{5}{4}-1}\frac{du}{4\lambda}\,d\lambda$$
$$= \frac{\Gamma\left(\frac{5}{4}\right)}{2^{\frac{1}{4}}}\int_0^{3^{\frac{3}{4}}} (w\nu * w\nu * w\nu)^2(1,t^{-4})\,dt.$$

Remark 2.2.6. As we saw in Proposition 2.2.4 the value of the convolution measure depends only on one parameter. This because $\nu * \nu * \nu$ is radial and it is constant along branches of the quartic $\tau = \alpha\xi^4$. Let $\alpha(t) = t^{-4}$ the amplitude of the quartic $\tau = \alpha(t)\xi^4$. When t ranges in $(0, 3^{\frac{3}{4}}]$, $\alpha(t)$ gives all possible amplitudes of quartic in the support of $\nu * \nu * \nu$.

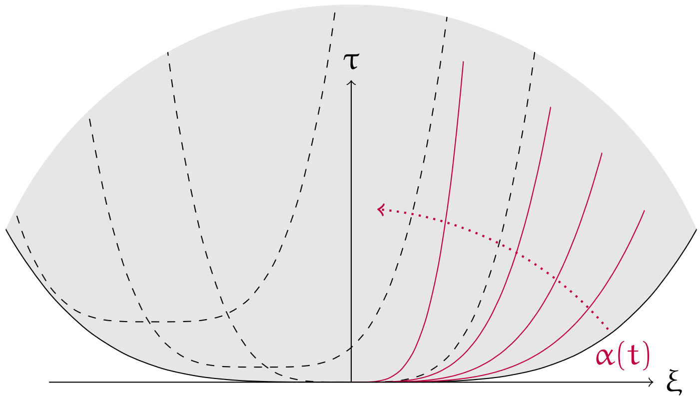


Figure 4.: Support of the measure $w\nu * w\nu * w\nu$. Its value in a point (ξ,τ) depends only on $\alpha(t)$.

After rescaling, since the integrand is even, we can write

$$\int_0^{3^{\frac{3}{4}}} (w\nu * w\nu * w\nu)^2(1,t^{-4})\,dt = 3^{\frac{3}{4}} \cdot \frac{1}{2}\int_{-1}^{1} g^2(t)\,dt$$

where the function of the right hand side is $g(t) = (w\nu * w\nu * w\nu)(1, 3^{-3}t^{-4})$. We have

$$\frac{\|f\sqrt{w\nu} * f\sqrt{w\nu} * f\sqrt{w\nu}\|^2_{L^2(\mathbb{R}^2)}}{\|f\|^6_{L^2(\mathbb{R})}} = \left(\frac{2^4 3^{\frac{3}{4}}\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{5}{12}\right)^3}\right)⨍_{-1}^{1} g^2(t)\,dt. \tag{2.2.10}$$

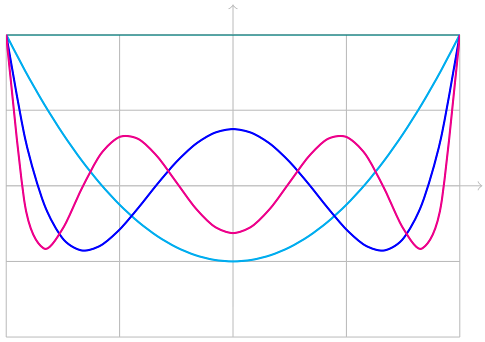
Thus, computing a lower bound for the numerator in the left hand side it reduces to approximate the norm of g in $L^2([-1,1])$ equipped with the normalised scalar product $\langle f,g\rangle = ⨍_{-1}^{1} fg$. With this purpose in mind, we consider the set of all monomials $\{1, t, t^2, \dots\}$. It is a complete system in $L^2([-1,1])$. Using the Gram-Schmidt process we obtain a well-known orthogonal basis: the Legendre polynomials.

Definition 2.2.7 (Legendre polynomials)**.** Let denote with $P_n = P_n(t)$ the solution to the differential equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right] + n(n+1)P_n(x) = 0.$$

The function $P_n(t)$ is the n^{th} Legendre polynomial. The set $\{P_n\}_{n\in\mathbb{N}}$ is an orthogonal basis of $L^2([-1,1])$.

The first four even Legendre polynomials are plotted below.



$P_0(t) = 1,$

$P_2(t) = \frac{1}{2}(3t^2 - 1),$

$P_4(t) = \frac{1}{8}(35t^4 - 30t^2 + 3),$

$P_6(t) = \frac{1}{16}(231t^6 - 315t^4 + 105t^2 - 5).$

Figure 5.: First four even Legendre polynomials.

Moreover, with the normalised scalar product, we have that

$$\langle P_m, P_n \rangle = \fint_{-1}^{1} P_m(t) P_n(t)\, dt = \frac{1}{2n+1} \delta_{m,n},$$

thus $\|P_n\|^2 = (2n+1)^{-1}$. We indicate with Q_n the normalised polynomial $\frac{P_n}{\|P_n\|}$. Then $Q_n = (\sqrt{2n+1})\, P_n$ and $\{Q_n\}_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2([-1,1], \langle \cdot, \cdot \rangle)$.

We are now ready to prove the existence of maximizers.

Proof of Theorem 2.2.1. The norm of g can be written as

$$\|g\|_{L^2}^2 = \sum_{n \geqslant 0} \langle g, Q_n \rangle^2 = \sum_{n \geqslant 0} \langle g, Q_{2n} \rangle^2,$$

since the function g is even. We can approximate the norm calculating arbitrarily many coefficients:

$$c_n^2 := \langle g, Q_n \rangle^2 = (2n+1) \left(\fint_{-1}^{1} g(t) P_n(t)\, dt \right)^2.$$

These coefficients can be retrieved from the *moments* of the measure g:

$$\mathcal{J}_k := \int_{-1}^{1} g(t)\, t^k\, dt,$$

once we have them, one can obtain the value of c_n.

To compute $\mathcal{J}_k$ we consider the function $f(x)\sqrt{w(x)} e^{bx} =: h_b^2(x)$. In view of (2.2.5) and (2.2.6) we have

$$\int_{\mathbb{R}^2} (h_b^2 \nu * h_b^2 \nu * h_b^2 \nu)(\xi, \tau)\, d\xi\, d\tau = \left(\|h_b\|_{L^2(\mathbb{R})}^2 \right)^3 =: G(b)^3$$
$$\int_{\mathbb{R}^2} (h_b^2 \nu * h_b^2 \nu * h_b^2 \nu)(\xi, \tau)\, d\xi\, d\tau = \int_{\mathbb{R}^2} e^{-(\tau - b\xi)} (w\nu * w\nu * w\nu)(\xi, \tau)\, d\xi\, d\tau =: F(b).$$

We write F and G as Taylor series. Then we expand the cube and rearrange the terms in series, we have:

$$F(b) = \sum_{n\geqslant 0} \frac{F^{(n)}(0)}{n!}\, b^n, \qquad G(b)^3 = \left(\sum_{n\geqslant 0} \frac{G^{(n)}(0)}{n!}\, b^n\right)^3 = \sum_{n\geqslant 0} \frac{d_n}{n!}\, b^n.$$

Thus, for every $n \in \mathbb{N}$

$$F^{(n)}(0) = d_n.$$

Also notice that, because of Proposition 2.2.4, the functions F and G are *even*. In particular in the above series only even coefficients appear.

$$F(b) = \sum_{n\geqslant 0} \frac{F^{(2n)}(0)}{(2n)!}\, b^{2n}, \qquad G(b)^3 = \sum_{n\geqslant 0} \frac{d_{2n}}{(2n)!}\, b^{2n}.$$

The first five coefficients of the second expansion are

$$d_0 = G(0)^3 \tag{2.2.11}$$

$$d_2 = 3\, G(0)^2 G''(0) \tag{2.2.12}$$

$$d_4 = 3\, G(0) \left(6\, G''(0)^2 + G(0) G^{(4)}(0)\right) \tag{2.2.13}$$

and $d_{2n+1} = 0$ for every $n \geqslant 0$.

We compute the derivatives:

$$\begin{aligned} G^{(2k)}(0) &= \int_{\mathbb{R}} e^{-x^4} (x^2)^{\frac{1}{3}} x^{2k}\, dx \\ &= 2\int_0^\infty e^{-x^4} x^{\frac{2}{3}+2k}\, dx \overset{(u=x^4)}{=} \frac{1}{2}\int_0^\infty e^{-u} u^{\frac{1}{6}+\frac{k}{2}} u^{\frac{1}{4}-1}\, du \\ &= \frac{1}{2}\int_0^\infty e^{-u} u^{\frac{5}{12}+\frac{k}{2}-1}\, du = \frac{1}{2}\Gamma\left(\frac{5}{12}+\frac{k}{2}\right). \end{aligned}$$

$$\begin{aligned} F^{(2k)}(0) &= \int_{\mathbb{R}^2} e^{-\tau} (w\nu * w\nu * w\nu)(\xi,\tau) \xi^{2k}\, d\tau\, d\xi \qquad (\text{changing variables: } \tau = \lambda\xi^4) \\ &= \int_{\frac{1}{27}}^\infty (w\nu * w\nu * w\nu)(1,\lambda) \left[2\int_0^\infty e^{-\lambda\xi^4} \xi^{4+2k}\, d\xi\right] d\lambda \\ &= \frac{1}{2}\Gamma\left(\frac{5}{4}+\frac{k}{2}\right) \int_{\frac{1}{27}}^\infty (w\nu * w\nu * w\nu)(1,\lambda) \frac{d\lambda}{\lambda^{\frac{5}{4}+\frac{k}{2}}} \\ &= \frac{1}{2}\Gamma\left(\frac{5}{4}+\frac{k}{2}\right) \int_0^{3^{\frac{3}{4}}} (w\nu * w\nu * w\nu)(1,t^{-4}) \frac{4\, dt}{t^{-2k}} \\ &= \Gamma\left(\frac{5}{4}+\frac{k}{2}\right) (3^{\frac{3}{4}})^{2k+1} \int_0^1 (w\nu * w\nu * w\nu)(1, 3^{-3} s^{-4}) s^{2k}\, ds \\ &= \Gamma\left(\frac{5}{4}+\frac{k}{2}\right) (3^{\frac{3}{4}})^{2k+1}\, \mathcal{J}_{2k}. \end{aligned}$$

where we changed variables back $\lambda^{-\frac{1}{4}} = t$, so that $\lambda^{-\frac{1}{4}-1}\, d\lambda = -4\, dt$, then we rescaled $t = 3^{\frac{3}{4}} s$.

The expressions for $F^{(2n)}(0)$ encapsulate the moments $\mathcal{I}_{2n}$. We compute $\mathcal{I}_{2n}$ comparing the coefficients $F^{(2n)}(0)$ with the explicit values of d_{2n}:

$$\mathcal{I}_{2n} = \frac{d_{2n}}{(3^{\frac{3}{4}})^{2n+1}\Gamma\left(\frac{5}{4}+\frac{n}{2}\right)}. \tag{2.2.14}$$

THE FIRST COEFFICIENT c_0 Because of the normalised product we have chosen, the double of the first coefficient c_0 equals the first moment $\mathcal{I}_0$. We compare the first coefficients of the two series using the formulas (2.2.14) and Eqs. (2.2.11) to (2.2.13).

$$\begin{cases} F(0) & = 3^{\frac{3}{4}}\Gamma(\frac{5}{4})\, 2\, c_0 \\ G(0)^3 & = 2^{-3}\left(\Gamma(\frac{5}{12})\right)^3 \end{cases} \quad \Rightarrow \quad c_0 = \frac{\left(\Gamma(\frac{5}{12})\right)^3}{2^4 3^{\frac{3}{4}}\Gamma(\frac{5}{4})}.$$

This first coefficient is an old friend. In fact, this quantity is closely related to the constant in front of the squared norm of g in (2.2.10): that constant is c_0^{-1}. Truncating the expansion at the first step, what we get is exactly our first approximation (2.2.7):

$$\frac{\| f\sqrt{w}\nu * f\sqrt{w}\nu * f\sqrt{w}\nu\|^2_{L^2(\mathbb{R}^2)}}{\|f\|^6_{L^2(\mathbb{R})}} \geqslant \frac{1}{c_0}\cdot c_0^2 = c_0$$

Now we can continue computing more coefficients.

THE COEFFICIENT c_2 To calculate c_2 we need the second moment $\mathcal{I}_2$. This is

$$\mathcal{I}_2 = \frac{3\, G(0)^2 G''(0)}{(3^{\frac{3}{4}})^3\,\Gamma(\frac{7}{4})} = \frac{3\,\Gamma(\frac{5}{12})^2\Gamma(\frac{11}{12})}{2^3 3^{\frac{9}{4}}\,\Gamma(\frac{7}{4})} = \alpha_2\, c_0, \qquad \alpha_2 := \frac{2\,\Gamma(\frac{11}{12})\Gamma(\frac{1}{4})}{3\sqrt{3}\,\Gamma(\frac{5}{12})\Gamma(\frac{3}{4})}.$$

The second coefficient squared is

$$c_2^2 = 5\langle g, \tfrac{1}{2}(3t^2-1)\rangle^2 = \tfrac{5}{4^2}(3\mathcal{I}_2 - \mathcal{I}_0)^2.$$

THE COEFFICIENT c_4 The fourth moment $\mathcal{I}_4$ is:

$$\mathcal{I}_4 = \left(\frac{2^4}{15}\frac{\Gamma(\frac{11}{12})^2}{\Gamma(\frac{5}{12})^2} + \frac{2}{3^3}\right) c_0 = \alpha_4\, c_0.$$

The fourth coefficient squared is

$$c_4^2 = 9\langle g, \tfrac{1}{8}(35t^4 - 30t^2 + 3)\rangle^2 = \tfrac{9}{4}\tfrac{1}{8^2}(35\mathcal{I}_4 - 30\mathcal{I}_2 + 3\mathcal{I}_0)^2.$$

We sum all the coefficients squared that we have calculated so far, in terms of c_0. After collecting c_0, we obtain:

$$c_0^2 + c_2^2 + c_4^2 = \left(\frac{9}{4} - \frac{15}{4}\alpha + \frac{45}{4^2}\alpha^2 + \frac{9}{2^8}(35\alpha_4 - 30\alpha_2 + 6)^2\right) c_0^2.$$

Since

$$\frac{c_0^2 + c_2^2 + c_4^2}{c_0} \approx 0.306879 > \frac{\pi}{6\sqrt{3}}$$

this proves a lower bound for $\mathbf{S}$ good enough to ensure that $\mathbf{S} > S_{\mathrm{Schr}}$, since

$$\frac{\pi}{3}\mathbf{S}^6 \geqslant \frac{\|f\sqrt{w}\nu * f\sqrt{w}\nu * f\sqrt{w}\nu\|^2_{L^2(\mathbb{R}^2)}}{\|f\|^6_{L^2(\mathbb{R})}} \geqslant \frac{c_0^2 + c_2^2 + c_4^2}{c_0} > \frac{\pi}{3}(S_{\mathrm{Schr}})^6$$

This rules out the second case in Theorem 2.2.2, proving existence of extremizers. □

REGULARITY OF EXTREMIZERS

In this section we study regularity of extremizers for the Strichartz estimate (2.1.3). We prove the following theorems.

Theorem 2.3.1. *Let be* f *an extremizer of* (2.1.3)*. Then there exists* $\mu \in \mathbb{R}_+$ *such that*

$$\int_{-\infty}^{\infty} |e^{\mu|\xi|^4}\widehat{f}(\xi)|\, d\xi < \infty.$$

Since decay in frequency translates into regularity in space, as a consequence of Theorem 2.3.1 we obtain the following result.

Theorem 2.3.2. *Any extremizer of the Strichartz inequality* (2.1.3) *is a* smooth *function. Moreover*

$$\sup_{x\in\mathbb{R}} |\partial^n f(x)| < +\infty \qquad \textit{for any } n \in \mathbb{N}.$$

We recall our operator $T(t)$ previously defined in (2.1.4) as

$$T(t)f(x) := \int_{\mathbb{R}} e^{ix\xi} w(\xi) e^{it\xi^4} \widehat{f}(\xi)\, đ\xi, \qquad w(\xi) = (6\xi^2)^{\frac{1}{6}}.$$

Euler-Lagrange equation

In the previous chapter we settled the existence of extremizers for the Strichartz inequality (2.1.3). Then the sharp constant **S** is a *maximum*:

$$\mathbf{S} := \max_{\substack{f\in L^2(\mathbb{R}),\\ f\neq 0}} \frac{\|T(t)f\|_{L^6(\mathbb{R}^2)}}{\|f\|_{L^2(\mathbb{R})}}.$$

We consider the functional $\mathfrak{L}\colon L^2(\mathbb{R}) \to \mathbb{R}$ defined as

$$\mathfrak{L}(f) := \frac{\|T(t)f\|^6_{L^6_{t,x}(\mathbb{R}^2)}}{\|f\|^6_{L^2(\mathbb{R})}}.$$

The extremizers of (2.1.3) are critical points of $\mathfrak{L}$, in particular they are solutions of the following equation.

$$\frac{\partial}{\partial\tau}\mathfrak{L}(f+\tau v) \restriction_{\tau=0} = 0, \quad \forall v \in L^2(\mathbb{R}).$$

We derive formally, exchanging the derivative and the integral.

$$\begin{aligned}\frac{\partial}{\partial\tau}|T(f+\tau v)|^6 \restriction_{\tau=0} &= 3|Tf|^4(Tv\overline{Tf} + \overline{Tv}Tf)\\ &= 6|Tf|^4\Re(Tv\overline{Tf})\\ &= 6|Tf|^4\Re(Tf\overline{Tv}).\end{aligned}$$

$$\frac{\partial}{\partial \tau}\|f+\tau v\|_{L^2}^6 \restriction_{\tau=0} = 6\|f\|_{L^2(\mathbb{R})}^4 \int_{\mathbb{R}} \mathfrak{R}(f\overline{v}).$$

We obtain the equation:

$$\iint_{\mathbb{R}^2} |Tf|^4 \mathfrak{R}(Tf\overline{Tv})\, dt\, dx = \underbrace{\frac{\|T(t)f\|_{L^6(\mathbb{R}^2)}^6}{\|f\|_{L^2(\mathbb{R})}^6}\|f\|_{L^2(\mathbb{R})}^4}_{\omega} \int_{\mathbb{R}} \mathfrak{R}(f\overline{v})\, dx.$$

The quantity ω is a Lagrange multiplier that equals $\mathbf{S}^6\|f\|_{L^2(\mathbb{R})}^4$ when f is an extremizer. Since $|Tf|^4$ is real, we can take out the real part in both sides. Then, any extremizer has to satisfy the equation

$$\mathfrak{R}\left(\iint_{\mathbb{R}^2} |Tf|^4 Tf\,\overline{Tv}\, dt\, dx\right) = \omega\, \mathfrak{R}\left(\int_{\mathbb{R}} f\,\overline{v}\, dx\right), \quad \forall v \in L^2(\mathbb{R}). \tag{2.3.1}$$

The left hand side is well defined for every $f, v \in L^2$. Expanding the integral Tv we have

$$Tv = \int w(\xi) e^{i(t\xi^4 + x\xi)} \int e^{-iy\xi} v(y)\, dy\, đ\xi$$

so we can rewrite

$$\overline{Tv} = \int_{\mathbb{R}_\xi} w(\xi) e^{-i(t\xi^4+x\xi)} \int_{\mathbb{R}_y} e^{iy\xi}\overline{v(y)}\, dy\, đ\xi =$$
$$= \int_{\mathbb{R}_y}\int_{\mathbb{R}_\xi} w(\xi) e^{-i(t\xi^4+x\xi)} e^{iy\xi}\,\overline{v(y)}\, đ\xi\, dy.$$

Denote with $u(x,t)$ the function $|Tf|^4 Tf$; the left hand side in (2.3.1) becomes

$$\int_{\mathbb{R}_y}\int_{\mathbb{R}_t}\int_{\mathbb{R}_\xi} e^{iy\xi} w(\xi) e^{-it\xi^4} \int_{\mathbb{R}_x} e^{-ix\xi}\, u(x,t)\, dx\, đ\xi\, dt\, \overline{v(y)}\, dy.$$

This equals

$$\int_{\mathbb{R}_y}\int_{\mathbb{R}_t} T^\star u\, dt\, \overline{v(y)}\, dy, \quad \text{that is} \quad \left\langle \int_{\mathbb{R}_t} T^\star |Tf|^4 Tf\, dt, v \right\rangle_{L_y^2}$$

where

$$T^\star u = \int_{\mathbb{R}_\xi} e^{iy\xi} w(\xi) e^{-it\xi^4} \mathcal{F}_x u(x,t)\, đ\xi = (w(\xi)e^{-it\xi^4}\widehat{u})\check{} = D_0^{\frac{1}{3}} e^{-it\triangle^2} u.$$

We have obtained that any extremizer satisfies the equation:

$$\mathfrak{R}\left\langle \int_{\mathbb{R}_t} T^\star |Tf|^4 Tf\, dt, v \right\rangle = \omega\, \mathfrak{R}\langle f, v\rangle, \quad \forall v \in L^2(\mathbb{R}). \tag{2.3.2}$$

This equation has to hold for all $v \in L^2(\mathbb{R})$. Consider iv, then the hermitian product $\langle \cdot, iv \rangle = -i\langle \cdot, v \rangle$, and $\mathfrak{R}\langle \cdot, iv \rangle = \mathfrak{R}(-i\langle \cdot, v \rangle) = \mathfrak{I}\langle \cdot, v \rangle$, so the above equality for the real part implies the one for the imaginary part. Thus we can forget about the real part and write:

$$\left\langle \int_{\mathbb{R}_t} T^\star |Tf|^4 Tf \, dt, v \right\rangle = \omega \, \langle f, v \rangle , \quad \forall \, v \in L^2(\mathbb{R}). \tag{weak E-L}$$

This means

$$\left(\int_{\mathbb{R}_t} T^\star |Tf|^4 Tf \, dt - \omega f \right) \perp v, \quad \forall \, v \in L^2(\mathbb{R})$$

and so it has to be the zero element in $L^2(\mathbb{R})$. We obtained the Euler-Lagrange equation of $\mathfrak{L}$:

$$\int_{\mathbb{R}_t} T^\star |Tf|^4 Tf \, dt = \omega \, f. \tag{E-L}$$

In particular, any extremizer $f \in L^2(\mathbb{R})$ of (2.1.3) satisfies the Euler-Lagrange equation.

We introduce also the following definition.

Definition 2.3.3 (Weak solutions)**.** A function f in $L^2(\mathbb{R})$ that is a solution to (weak E-L) is called *weak solution* of the Euler-Lagrange equation.

Motivated by these formulas, we introduce the 6-linear form

$$Q(f_1, f_2, f_3, f_4, f_5, f_6) := \iint_{\mathbb{R}^2} \prod_{i=1}^{3} Tf_i \prod_{i=4}^{6} \overline{Tf_i} \, dt \, dx. \tag{2.3.3}$$

Remark 2.3.4. Notice that $Q(f, f, f, f, f, v)$ equals the left hand side of (weak E-L). Thus, when f is a weak solution, we have

$$\omega \, \langle v, f \rangle = Q(v, f, f, f, f, f), \quad \forall \, v \in L^2(\mathbb{R}). \tag{2.3.4}$$

Moreover, when all the arguments coincide, $Q(f, f, f, f, f, f) = \|Tf\|_6^6$.

The Strichartz inequality (2.1.3) gives a bound for the 6-linear form.

Lemma 2.3.5. *For* $i = 1, \dots, 6$ *let* f_i *be functions in* $L^2(\mathbb{R})$*. Then*

$$|Q(f_1, f_2, f_3, f_4, f_5, f_6)| \lesssim \prod_{i=1}^{6} \|f_i\|_{L^2}.$$

Proof. The bound follows by applying generalised Hölder inequality and the Strichartz estimate for the propagator T:

$$|Q(f_1, f_2, f_3, f_4, f_5, f_6)| \leqslant \int \prod_{i=1}^{6} |Tf_i| \, dt \, dx \leqslant \prod_{i=1}^{6} \|Tf_i\|_{L^6(\mathbb{R}^2)} \leqslant \mathbf{S}^6 \prod_{i=1}^{6} \|f_i\|_{L^2}.$$

□

Expanding the definition of the operator T in (2.3.3), we rewrite the form Q as

$$Q(f_1,f_2,f_3,f_4,f_5,f_6) = \iint_{\mathbb{R}^2}\int_{\mathbb{R}^6} e^{ita(\xi)+ixb(\xi)} \prod_{\substack{j=1,2,3\\ l=4,5,6}} \widehat{f}_j(\xi_j)\overline{\widehat{f}_l(\xi_l)} \prod_1^6 w(\xi_j)\, d\!\!\bar{}\,\xi_j \, dt\, dx$$

$$= \frac{1}{(2\pi)^4}\int_{\mathbb{R}^6} \delta\big(a(\xi)\big)\,\delta\big(b(\xi)\big) \prod_{j=1,2,3} \widehat{f}_j(\xi_j)\overline{\widehat{f}_{j+3}(\xi_{j+3})} \prod_1^6 w(\xi_j)\, d\xi_j$$

where ξ is the vector $(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)$ and the functions a, b are defined as

$$a(\xi) := \sum_{i=1,2,3} \xi_i^4 - \xi_{i+3}^4 \qquad , \qquad b(\xi) := \sum_{i=1,2,3} \xi_i - \xi_{i+3}.$$

We also used the formula $\delta(k) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iks}\, ds$. See Appendix A, and in particular (A.0.5).

Remark 2.3.6. Using the above expression for Q, we have

$$|Q(f_1,\dots,f_6)| \leqslant \frac{1}{(2\pi)^4}\int_{\mathbb{R}^6} \delta\big(a(\xi)\big)\,\delta\big(b(\xi)\big) \prod_{j=1,\dots,6} |\widehat{f}_j(\xi_j)| w(\xi_j)\, d\xi_j = Q(|\widehat{f}_1|\check{\ },\dots,|\widehat{f}_6|\check{\ }).$$

For simplicity, we define

$$\check{Q}(\,\cdot\,,\dots,\,\cdot\,) := Q(|\cdot|\check{\ },\dots,|\cdot|\check{\ }). \tag{2.3.5}$$

We introduce also the exponentially weighted form, that we will use later. Given a nonnegative function $F(x)$, we denote

$$\check{Q}_F(f_1,f_2,f_3,f_4,f_5,f_6) = \check{Q}(e^F f_1, e^{-F}f_2, e^{-F}f_3, e^{-F}f_4, e^{-F}f_5, e^{-F}f_6).$$

In particular

$$\check{Q}_F(h_1,h_2,h_3,h_4,h_5,h_6) = Q((e^F|h_1|)\check{\ },(e^{-F}|h_2|)\check{\ },(e^{-F}|h_3|)\check{\ },\dots,(e^{-F}|h_6|)\check{\ })$$

$$= \frac{1}{(2\pi)^4}\int_{\mathbb{R}^6} \delta\big(a(\xi)\big)\,\delta\big(b(\xi)\big)\, e^{F(\xi_1)-\sum_2^6 F(\xi_j)} \prod_{j=1}^6 |h_j(\xi_j)|\, d\xi_j.$$

Bilinear estimates

Using generalised Hölder inequality with $p = 3$, we can split the product in a different way.

$$|Q((f_i)_{i=1,\dots,6})| \leqslant \int \prod_{i=1}^6 |Tf_i|\, dt\, dx \leqslant \|Tf_1Tf_2\|_{L^3}\, \|Tf_3Tf_4\|_{L^3}\, \|Tf_5Tf_6\|_{L^3}$$

$$\leqslant \|Tf_1Tf_2\|_{L^3} \quad \mathbf{S}^4 \prod_{i=3}^6 \|f_i\|_{L^2}.$$

Up to permutation, we can consider any pair f_i, f_j above and estimate $\|Tf_iTf_j\|_{L^3}$. We can gain some decay if, for some distinct i and j, f_i and f_j have "well separated" support in frequency. This is a well-known result for the free Schrödinger propagator $e^{it\triangle}$ (see [HL09]), where one has the following

Theorem 2.3.7 (Fourier bilinear estimate for free Schrödinger). *Let* $f_1, f_2 \in L^2(\mathbb{R})$ *and let* $c = \operatorname{dist}(\operatorname{supp}\widehat{f_1}, \operatorname{supp}\widehat{f_2}) > 0$. *Then*

$$\|e^{it\triangle}f_1 e^{it\triangle}f_2\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R})} \leqslant \frac{1}{\sqrt{2c}}\|f_1\|_{L^2(\mathbb{R})}\,\|f_2\|_{L^2(\mathbb{R})}.$$

We prove a similar result for our operator $T(t)$. Due to the positive weights $w(\xi) \sim |\xi|^{\frac{1}{3}}$, the decay is not directly related to the distance of the support in frequency, but to the ratio of the closest endpoints of the two supports. Also the $\triangle^2$ gives a different power in the decay of the Jacobian.
The choice of support is meant for exploiting the Littlewood-Paley decomposition later, as in [HS12, Lemma 4.1]. See Appendix A.1 for an introduction to the Littlewood-Paley theory.

Lemma 2.3.8 (Fourier bilinear estimate). *Suppose that exist two distinct* $i, j \in \{1, \ldots, 6\}$, *and* $N_1 \ll N_2$ *such that*

$$\operatorname{supp}\widehat{f_i} \subseteq \{\xi : |\xi| \leqslant N_1\} \quad \textit{and} \quad \operatorname{supp}\widehat{f_j} \subseteq \{\xi : N_2 \leqslant |\xi| \leqslant 2N_2\}.$$

Then there exists a constant $C > 0$ *independent of* N_1, N_2 *and* f_i, f_j *such that*

$$\|Tf_iTf_j\|_{L^3_{t,x}(\mathbb{R}^2)} \leqslant C\sqrt[3]{\frac{N_1}{N_2^2}}\,\|f_i\|_{L^2(\mathbb{R})}\,\|f_j\|_{L^2(\mathbb{R})}.$$

Proof. For the sake of clarity, we assume that the hypothesis are satisfied for $i = 1$ and $j = 2$. We want to estimate

$$\|Tf_1Tf_2\|_{L^3} = \Big\|\int e^{it(\xi_1^4+\xi_2^4)+ix(\xi_1+\xi_2)}w(\xi_1)w(\xi_2)\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\,đ\xi_1\,đ\xi_2\Big\|_{L^3}.$$

Changing variables and indicating with Jac the Jacobian

$$\begin{cases} a & := \xi_1^4 + \xi_2^4 \\ b & := \xi_1 + \xi_2 \end{cases} \quad \text{and} \quad \operatorname{Jac}(\xi_1, \xi_2) = \begin{pmatrix} 4\xi_1^3 & 4\xi_2^3 \\ 1 & 1 \end{pmatrix}.$$

Integrating where $\xi_1 \in [0, N_1]$ and $\xi_2 \in [N_2, 2N_2]$, since $N_2 \gg N_1$, we have

$$|\det \operatorname{Jac}| = 4|\xi_2^3 - \xi_1^3| = |\xi_2|^3\, 4\left(1 - \left(\frac{\xi_1}{\xi_2}\right)^3\right) \sim N_2^3.$$

Our integral becomes

$$\int e^{ita+ixb} w(\xi_1)w(\xi_2)\widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)\frac{đa\, đb}{|\det \mathrm{Jac}|} = \mathcal{F}\left(\frac{w(\xi_1)w(\xi_2)\widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)}{|\det \mathrm{Jac}|}\right)(-x,-t)$$

where ξ_1, ξ_2 are now functions of a and b and $\mathcal{F}$ is the Fourier transform in (x,t).

We use Hausdorff-Young inequality

$$\left\|\mathcal{F}\left((w\otimes w\cdot \widehat{f}_1\otimes\widehat{f}_2)\,|\det \mathrm{Jac}|^{-1}\right)\right\|_{L^3_{t,x}(\mathbb{R}^2)} \leqslant \|(w\otimes w\cdot \widehat{f}_1\otimes\widehat{f}_2)\,|\det \mathrm{Jac}|^{-1}\|_{L^{3/2}(\mathbb{R}^2)}.$$

Then the right hand side equals

$$\left(\int_{\mathbb{R}_+\times\mathbb{R}} |\det \mathrm{Jac}|^{-\frac{1}{2}}\, |w(\xi_1(a,b))w(\xi_2(a,b))\widehat{f}_1(\xi_1(a,b))\widehat{f}_2(\xi_2(a,b))|^{\frac{3}{2}}\frac{đa\, đb}{|\det \mathrm{Jac}|}\right)^{\frac{2}{3}},$$

changing variables back we have

$$\left(\int_{\mathbb{R}\times\mathbb{R}} |\det \mathrm{Jac}|^{-\frac{1}{2}}\, |w(\xi_1)w(\xi_2)\widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)|^{\frac{3}{2}}\, đ\xi_1\, đ\xi_2\right)^{\frac{2}{3}}$$
$$\leqslant \frac{1}{(4|N_2^3-N_1^3|)^{\frac{1}{3}}}\left(\iint_{\mathbb{R}^2} |\widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)|^{\frac{3}{2}}\chi_1(\xi_1)\chi_2(\xi_2)\, d\xi_1\, d\xi_2\right)^{\frac{2}{3}}$$

where χ_1 and χ_2 are given by

$$\chi_1(\xi_1) = \frac{|\xi_1|^{\frac{1}{2}}}{2\pi}\,\mathbb{1}_{[0,N_1]}(|\xi_1|)$$
$$\chi_2(\xi_2) = \frac{|\xi_2|^{\frac{1}{2}}}{2\pi}\,\mathbb{1}_{[N_2,2N_2]}(|\xi_2|),$$

since the weight $w(\xi) \sim |\xi|^{\frac{1}{3}}$.

Using Hölder inequality with $(p,p') = (\frac{4}{3},4)$ we obtain

$$\|Tf_1 Tf_2\|_{L^3} \lesssim \frac{1}{N_2}\,\|\widehat{f}_1\|_2\,\|\chi_1\|_4^{\frac{2}{3}}\,\|\widehat{f}_2\|_2\,\|\chi_2\|_4^{\frac{2}{3}}$$
$$\lesssim \frac{1}{N_2}\,\sqrt[3]{N_1}\,\sqrt[3]{N_2}\,\|\widehat{f}_1\|_2\,\|\widehat{f}_2\|_2 = \left(\frac{N_1}{N_2^2}\right)^{\frac{1}{3}}\|f_1\|_2\,\|f_2\|_2.$$

This concludes the proof. □

Remark 2.3.9. In the previous lemma the presence of the weight $w(\xi) \sim |\xi|^{\frac{1}{3}}$ is not an issue, since we are assuming that $\widehat{f}_1$ and $\widehat{f}_2$ have *compact* support.

We can use this result to get decay even after dropping the compact support assumption on one of the functions.

Corollary 2.3.10. *Assume that there exist two distinct* $i, j \in \{1, \dots, 6\}$ *for which*

$$\operatorname{supp} \widehat{f_i} \subseteq \{\xi : |\xi| \leqslant s\} \quad \textit{and} \quad \operatorname{supp} \widehat{f_j} \subseteq \{\xi : |\xi| \geqslant Ns\}$$

for some $s > 1$*, and* $N \gg 1$*. Then there exists* $C > 0$ *independent of* s, N *and* f_i, f_j *such that*

$$\|Tf_i Tf_j\|_{L^3_{t,x}} \leqslant C \sqrt[3]{\frac{s}{N^2}} \|f_i\|_2 \|f_j\|_2.$$

Proof. Assume again $i = 1$ and $j = 2$. For $k \in \mathbb{Z}$, let be P_k the Littlewood-Paley projector on $A_k = \{\xi : 2^k \leqslant |\xi| < 2^{k+1}\}$ defined as $\widehat{P_k f} = \mathbb{1}_{A_k} \widehat{f}$. We can decompose f_2 as $\sum_{k: 2^{k+1} \geqslant Ns} P_k f_2$. Then, by triangle inequality

$$\begin{aligned}
\|Tf_1\, Tf_2\|_{L^3_{t,x}} &\leqslant \sum_{k: 2^{k+1} \geqslant Ns} \|Tf_1\, T(P_k f_2)\|_{L^3} \\
\text{applying Lemma 2.3.8} \quad &\lesssim \sum_{k: 2^{k+1} \geqslant Ns} \left(\frac{s}{2^{2k}}\right)^{\frac{1}{3}} \|f_1\|_2 \|P_k f_2\|_2 \\
&\leqslant \|f_1\|_2 \sqrt[3]{s} \sum_{k: 2^{k+1} \geqslant Ns} 2^{-2k/3} \|P_k f_2\|_2 \\
\text{applying Cauchy-Schwarz} \quad &\leqslant \|f_1\|_2 \sqrt[3]{s} \left(\sum_{k: 2^{k+1} \geqslant Ns} 2^{-\frac{4}{3}k}\right)^{\frac{1}{2}} \left(\sum_{k \in \mathbb{Z}} \|P_k f_2\|_2^2\right)^{\frac{1}{2}} \\
\text{using (A.1.1)} \quad &\lesssim \|f_1\|_2 \sqrt[3]{s}\, (Ns)^{-\frac{2}{3}} \|f_2\|_2 \\
&= \frac{\sqrt[3]{s}}{\sqrt[3]{N^2}} \|f_1\|_2 \|f_2\|_2.
\end{aligned}$$

This concludes the proof. □

Remark 2.3.11. The method used in the proof of Lemma 2.3.8 goes back to Carleson and Sjölin in the '70s [CS72] and it is a key ingredient for estimating the tails of the Fourier Transform of the extremizers. Since asymptotic properties of the Fourier transform $\widehat{f}$ translate in local properties of f in space, a super-polynomial decay of the tails in frequency will imply a (super) regularity in space: smoothness.

Remark 2.3.12. In the case of the free Schrödinger propagator $e^{it\Delta}$, bilinear estimates in frequency translate into estimates in space. This is due to the special multiplier associated to the free propagator, for which is possible to write explicitly the inverse Fourier transform:

$$e^{it\Delta} f(x) = \int_{\mathbb{R}} e^{ix\xi + it\xi^2} \widehat{f}(\xi)\, đ\xi = \left(e^{it\xi^2} \widehat{f}(\xi)\right)^{\vee} = \frac{1}{2\pi} \left(e^{it\xi^2}\right)^{\vee} * f = C \frac{1}{\sqrt{t}} \int_{\mathbb{R}} e^{-i\frac{|x-y|^2}{4t}} f(y)\, dy,$$

with $C > 0$.

We want to show that for any extremizer f in $L^2(\mathbb{R})$, there exists $\mu \in \mathbb{R}_+$ such that

$$\|e^{\mu|\cdot|}\,\widehat{f}\|_{L^1(\mathbb{R})} < +\infty. \tag{2.3.6}$$

One can also show a super exponential decay, proving that $e^{\mu|\xi|^n}\,\widehat{f}(\xi)$ belongs to $L^2(\mathbb{R})$ for some $n > 1$. (We will use later, for our case, $n = 4$.)

Lemma 2.3.13. *Assume that* $e^{\mu_0|\xi|^n}\,\widehat{f}(\xi) \in L^2(\mathbb{R})$ *for some* $\mu_0 > 0$ *and* $n > 1$, *then* $e^{\mu|\xi|}\widehat{f}(\xi) \in L^1(\mathbb{R})$ *for any* $\mu < \mu_0$.

Proof. We have that

$$e^{\mu|\xi|}\,\widehat{f}(\xi) = e^{\mu|\xi|-\mu_0|\xi|^n}\, e^{\mu_0|\xi|^n}\widehat{f}(\xi).$$

Applying Cauchy-Schwarz one has

$$\int e^{\mu|\xi|}|\widehat{f}(\xi)|\,d\xi = \int e^{\mu|\xi|-\mu_0|\xi|^n}\, e^{\mu_0|\xi|^n}|\widehat{f}(\xi)|\,d\xi \leqslant \|e^{\mu|\xi|-\mu_0|\xi|^n}\|_2\,\|e^{\mu_0|\xi|^n}\widehat{f}\|_2.$$

The second factor is bounded by hypothesis; the first one is finite, since

$$\|e^{\mu|\xi|-\mu_0|\xi|^n}\|_2^2 \lesssim \int_0^\infty e^{\mu\xi-\mu_0\xi^n}\,d\xi \leqslant \sup_{[0,1]}|e^{\mu\xi-\mu_0\xi^n}| + \int_1^\infty e^{(\mu-\mu_0)\xi^n} < \infty.$$

We showed the desired bound (2.3.6). □

Thus, as in [HS12], for the argument of the exponential we are allowed to use a particular class of functions which fits well into our problem.

Definition 2.3.14. Consider the function

$$F_{\mu,\varepsilon}(x) = \frac{\mu|x|^4}{1+\varepsilon|x|^4}, \qquad \text{for } \mu, \varepsilon > 0.$$

Notice that $F_{\mu,\varepsilon}(x) \to \mu|x|^4$ as $\varepsilon \to 0^+$.

Following the approach in [EHL11], our bound (2.3.6) will follow from an *uniform bound* in ε of $\|e^{F_{\mu,\varepsilon}(\xi)}\,\widehat{f}(\xi)\|_2$.

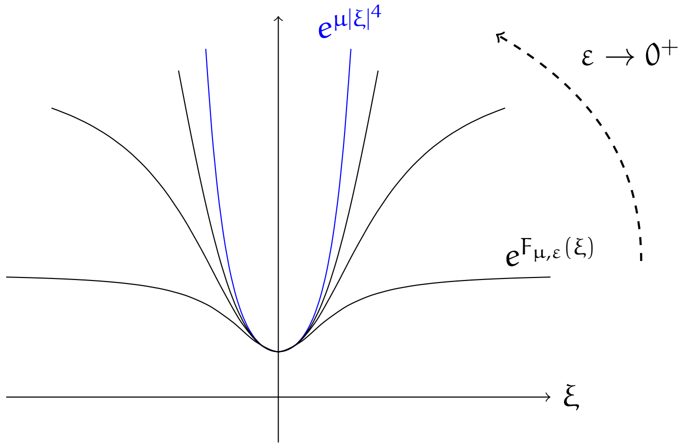


Figure 6.: Plot of the functions $e^{F_{\mu,\varepsilon}}$ as ε approaches zero.

With this choice of F we are able to control the exponentially weighted form

$$\check{Q}_{F_{\mu,\varepsilon}}(f_1,f_2,f_3,f_4,f_5,f_6) := \check{Q}(e^{F_{\mu,\varepsilon}}f_1, e^{-F_{\mu,\varepsilon}}f_2, e^{-F_{\mu,\varepsilon}}f_3, e^{-F_{\mu,\varepsilon}}f_4, e^{-F_{\mu,\varepsilon}}f_5, e^{-F_{\mu,\varepsilon}}f_6).$$

Proposition 2.3.15. *Let* h_j *be in* $L^2(\mathbb{R})$ *for* $j = 1, \dots, 6$. *Then for every* $\mu, \varepsilon \geqslant 0$

$$\check{Q}_{F_{\mu,\varepsilon}}(h_1, \dots, h_6) \leqslant \check{Q}(h_1, \dots, h_6).$$

Proof. Since

$$\check{Q}_{F_{\mu,\varepsilon}}(h_1, \dots, h_6) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^6} \delta\big(a(\xi)\big)\, \delta\big(b(\xi)\big)\, e^{F_{\mu,\varepsilon}(\xi_1) - \sum_2^6 F_{\mu,\varepsilon}(\xi_j)} \prod_{j=1}^{6} |h_j(\xi_j)|\, d\xi_j,$$

we want to prove that $e^{F_{\mu,\varepsilon}(\xi_1) - \sum_2^6 F_{\mu,\varepsilon}(\xi_j)} \leqslant 1$, so it is enough to show

$$F_{\mu,\varepsilon}(\xi_1) \leqslant \sum_{j=2}^{6} F_{\mu,\varepsilon}(\xi_j).$$

Under the hypothesis $a(\xi) = 0$, we have $\xi_1^4 = -\xi_2^4 - \xi_3^4 + \xi_4^4 + \xi_5^4 + \xi_6^4$ and in particular $\xi_1^4 \leqslant \sum_{j=2}^6 \xi_j^4$.

The function

$$t \mapsto \frac{\mu t}{1 + \varepsilon t}$$

is increasing on $\mathbb{R}_+$ for every positive μ, ε.

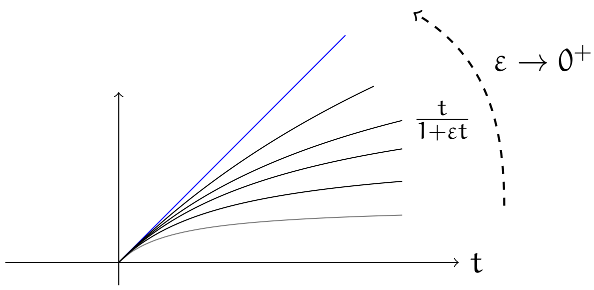


Figure 7.: Plot of the functions $t \mapsto \frac{t}{1+\varepsilon t}$ for different values of $\varepsilon \in [0, 1)$.

Then

$$\begin{aligned} F_{\mu,\varepsilon}(\xi_1) = \frac{\mu \xi_1^4}{1 + \varepsilon \xi_1^4} &\leqslant \mu \frac{\sum_2^6 \xi_j^4}{1 + \varepsilon \sum_2^6 \xi_k^4} \\ &= \sum_{j=2}^{6} \frac{\mu \xi_j^4}{1 + \varepsilon \sum_2^6 \xi_k^4} \leqslant \sum_{j=2}^{6} \frac{\mu \xi_j^4}{1 + \varepsilon \xi_j^4} = \sum_{j=2}^{6} F_{\mu,\varepsilon}(\xi_j) \end{aligned}$$

where in the last inequality we used $\sum_2^6 \xi_k^4 \geqslant \xi_j^4$ for some $j \in \{2, \dots, 6\}$. This concludes the proof. □

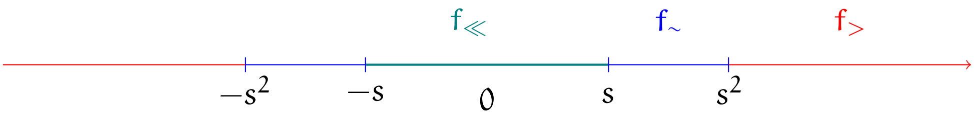
Let

$$\| f \|_{\mu,\varepsilon} := \| e^{F_{\mu,\varepsilon}} f \|_2.$$

We want to control $\| \widehat{f} \|_{\mu,\varepsilon}$.

Notation 2.3.16. Fix $s > 1$. For a function f we define

$$f_{\ll} := f \mathbb{1}_{[-s,s]}\,, \quad f_{<} := f \mathbb{1}_{[-s^2,s^2]}\,, \quad f_{>} := f \mathbb{1}_{[-s^2,s^2]^c}\,, \quad f_{\sim} := f_{<} - f_{\ll}\,. \qquad (2.3.7)$$



Recall that every extremizer f is also a weak solution of the Euler-Lagrange equation (Definition 2.3.3).

Lemma 2.3.17. *Let f be a weak solution of the (weak E-L), with $\omega > 0$ and $\|f\|_2 = 1$. Then, for every fixed $s > 1$, taking $\mu = 1/s^8$ we have*

$$\omega \|\widehat{f}_{>}\|_{\mu,\varepsilon} \lesssim \sum_{l=2}^{5} \|\widehat{f}_{>}\|_{\mu,\varepsilon}^{l} + \|\widehat{f}_{>}\|_{\mu,\varepsilon} \left(\frac{e^{s^{-4}}}{\sqrt[3]{s}} + \|\widehat{f}_{\sim}\|_2 \right) + \left(\frac{e^{s^{-4}}}{\sqrt[3]{s}} + \|\widehat{f}_{\sim}\|_2 \right),$$

where the implicit constant does not *depend on ε nor s.*

Proof. Let be $h = e^{F_{\mu,\varepsilon}} \widehat{f}$, so that $\|\widehat{f}\|_{\mu,\varepsilon} = \|h\|_2$. Exploiting the definition of weak solution, in particular (2.3.4), we have

$$\omega \|\widehat{f}_{>}\|_{\mu,\varepsilon}^2 = \omega \|e^{F_{\mu,\varepsilon}} \widehat{f}_{>}\|_2^2 = \omega \langle e^{2F_{\mu,\varepsilon}} \widehat{f}_{>}, \widehat{f} \rangle = \omega \langle (e^{2F_{\mu,\varepsilon}} \widehat{f}_{>})^{\vee}, f \rangle = Q((e^{2F_{\mu,\varepsilon}} \widehat{f}_{>})^{\vee}, f, \dots, f)$$

and

$$Q((e^{2F_{\mu,\varepsilon}} \widehat{f}_{>})^{\vee}, f, \dots, f) \leqslant Q((e^{F_{\mu,\varepsilon}} |h_{>}|)^{\vee}, (e^{-F_{\mu,\varepsilon}} |h|)^{\vee}, \dots, (e^{-F_{\mu,\varepsilon}} |h|)^{\vee}) = \check{Q}_{F_{\mu,\varepsilon}}(h_{>}, h, \dots, h).$$

By Proposition 2.3.15, we have

$$\check{Q}_{F_{\mu,\varepsilon}}(h_{>}, h, \dots, h) \leqslant \check{Q}(h_{>}, h, \dots, h).$$

We split h as in (2.3.7). By sublinearity of $Q(|\cdot|, \dots, |\cdot|)$

$$\begin{aligned} \check{Q}(h_{>}, h, \dots, h) &\leqslant \check{Q}(h_{>}, h_{<}, h, h, h, h) + \check{Q}(h_{>}, h_{>}, h, h, h, h) \\ &\lesssim \check{Q}(h_{>}, h_{<}, h_{<}, h_{<}, h_{<}, h_{<}) && \text{(I)} \\ &\quad + \check{Q}(h_{>}, h_{>}, h_{<}, h_{<}, h_{<}, h_{<}) && \text{(II)} \\ &\quad + \sum \check{Q}(h_{>}, h_{j_1}, h_{j_2}, h_{j_3}, h_{j_4}, h_{j_5}). && \text{(III)} \end{aligned}$$

The last sum is over all other possible combination with at least two j_i equal to $>$.
Term: (I) We split further writing $h_{<} = h_{\ll} + h_{\sim}$. By sublinearity again

$$Q(h_{>}, h_{<}, h_{<}, h_{<}, h_{<}, h_{<}) \leqslant Q(h_{>}, h_{\ll}, h_{<}, h_{<}, h_{<}, h_{<}) + Q(h_{>}, h_{\sim}, h_{<}, h_{<}, h_{<}, h_{<}).$$

Using bilinear estimate as in Corollary 2.3.10

$$\begin{aligned} Q(h_{>}, h_{\ll}, h_{<}, h_{<}, h_{<}, h_{<}) &\lesssim \frac{1}{\sqrt[3]{s}} \|h_{>}\|_2 \|h_{\ll}\|_2 \, \|h_{<}\|_2^4, \\ Q(h_{>}, h_{\sim}, h_{<}, h_{<}, h_{<}, h_{<}) &\lesssim \|h_{>}\|_2 \|h_{\sim}\|_2 \, \|h_{<}\|_2^4. \end{aligned}$$

Recalling that $\|f\|_2 = 1$, we have

$$\|h_<\|_2 = \|e^{F_{\mu,\varepsilon}}\widehat{f}_<\|_2 \leqslant \|e^{\mu|x|^4}\widehat{f}_<\|_2 \leqslant \sup_{[-s^2,s^2]} |e^{\mu x^4}|\,\|\widehat{f}\|_2 = e^{\mu s^8} \tag{2.3.8}$$
$$\|h_\ll\|_2 = \|e^{F_{\mu,\varepsilon}}\widehat{f}_\ll\|_2 \leqslant \|e^{\mu|x|^4}\widehat{f}_\ll\|_2 \leqslant \sup_{[-s,s]} |e^{\mu x^4}|\,\|\widehat{f}\|_2 = e^{\mu s^4}$$
$$\|h_\sim\|_2 \leqslant \|e^{\mu|x|^4}\widehat{f}_\sim\|_2 \leqslant \sup_{[-s^2,s^2]\setminus[-s,s]} |e^{\mu x^4}|\,\|\widehat{f}_\sim\|_2 = e^{\mu s^8}\,\|\widehat{f}_\sim\|_2.$$

Thus (I) is bounded by

$$Q(h_>, h_<, \ldots, h_<) \lesssim \|h_>\|_2 \left(\frac{e^{\mu(s^4-s^8)}}{\sqrt[3]{s}} + \|\widehat{f}_\sim\|_2\right) e^{5\mu s^8}.$$

Term (II) This case is similar to the previous one. By sublinearity

$$Q(h_>, h_>, h_<, h_<, h_<, h_<) \leqslant Q(h_>, h_>, h_\ll, , h_<, h_<, h_<) + Q(h_>, h_>, h_\sim, h_<, h_<, h_<).$$

Then use bilinear estimate as before

$$Q(h_>, h_>, h_\ll, h_<, h_<, h_<) \lesssim \|h_>\|_2 \frac{1}{\sqrt[3]{s}} \|h_>\|_2\|h_\ll\|_2\,\|h_<\|_2^3$$
$$Q(h_>, h_>, h_\sim, h_<, h_<, h_<) \lesssim \|h_>\|_2^2\,\|h_\sim\|_2\,\|h_<\|_2^3.$$

We obtain

$$Q(h_>, h_>, h_<, \ldots, h_<) \lesssim \|h_>\|_2^2 \left(\frac{e^{\mu(s^4-s^8)}}{\sqrt[3]{s}} + \|\widehat{f}_\sim\|_2\right) e^{4\mu s^8}.$$

Term: (III) Consider for example $Q(h_>, h_>, h_>, h_<, h_<, h_<)$, we have

$$Q(h_>, h_>, h_>, h_<, h_<, h_<) \lesssim \|h_>\|_2^3\,\|h_<\|_2^3 \leqslant e^{3\mu s^8}\,\|h_>\|_2^3,$$

where we used the bound for $\|h_<\|_2$ in (2.3.8).

Summing up, we have

$$\omega\|\widehat{f}_>\|_{\mu,\varepsilon}^2 \lesssim \|h_>\|_2 \left(\frac{e^{\mu(s^4-s^8)}}{\sqrt[3]{s}} + \|\widehat{f}_\sim\|_2\right) e^{5\mu s^8}$$
$$+ \|h_>\|_2^2 \left(\frac{e^{\mu(s^4-s^8)}}{\sqrt[3]{s}} + \|\widehat{f}_\sim\|_2\right) e^{4\mu s^8} + \|h_>\|_2\, e^{3\mu s^8} \sum_{l=3}^{5} \|h_>\|_2^l.$$

Dividing by $\|h_>\|_2$, recalling that $\|\widehat{f}_>\|_{\mu,\varepsilon} = \|h_>\|_2$, we get

$$\omega\|\widehat{f}_>\|_{\mu,\varepsilon} \lesssim \left(\frac{e^{\mu(s^4-s^8)}}{\sqrt[3]{s}} + \|\widehat{f}_\sim\|_2\right) e^{5\mu s^8} + \|h_>\|_2 \left(\frac{e^{\mu(s^4-s^8)}}{\sqrt[3]{s}} + \|\widehat{f}_\sim\|_2\right) e^{4\mu s^8} + e^{3\mu s^8} \sum_{l=3}^{5} \|h_>\|_2^l.$$

The norm $\|\widehat{f}_\sim\|_2 \leqslant \|\widehat{f}\,\mathbb{1}_{[-s,s]^c}\|_2 \to 0$ as $s \to \infty$. Taking $\mu = 1/s^8$

$$\frac{e^{\mu(s^4-s^8)}}{\sqrt[3]{s}} \sim \frac{e^{1/s^4}}{\sqrt[3]{s}} \to 0 \qquad \text{as } s \to \infty.$$

Then

$$\omega\| \widehat{f}_> \|_{\mu,\varepsilon} \lesssim \sum_{l=2}^{5} \| \widehat{f}_> \|_{\mu,\varepsilon}^{j} + \| \widehat{f}_> \|_{\mu,\varepsilon} \left(\frac{e^{s^{-4}}}{\sqrt[3]{s}} + \|\widehat{f}_\sim\|_2\right) + \left(\frac{e^{s^{-4}}}{\sqrt[3]{s}} + \|\widehat{f}_\sim\|_2\right).$$

This concludes the proof. □

We are now ready to prove Theorem 2.3.1.

Proof of Theorem 2.3.1. Call $\nu = \nu(s,\mu,\varepsilon) := \|\widehat{f}_>\|_{\mu,\varepsilon}$. Our aim is to prove that there exist $s_0 > 1$ and $\mu_0 > 0$ such that $\nu(s_0,\mu_0,0) < +\infty$.

Choosing $\mu = 1/s^8$, by Lemma 2.3.17, we have

$$\omega\nu \leqslant C\left(\sum_{j=2}^{5} \nu^j + \nu\left(\frac{e^{s^{-4}}}{\sqrt[3]{s}} + \|\widehat{f}_\sim\|_2\right) + \left(\frac{e^{s^{-4}}}{\sqrt[3]{s}} + \|\widehat{f}_\sim\|_2\right)\right), \tag{2.3.9}$$

where the constant C does not depend on s, nor on ε.
Notice that, with this choice of μ, the function ν only depends on s and ε. Furthermore, for fixed ε, ν goes to zero as s goes to infinity[2]. In particular, for $s > 1$, for any $\varepsilon \geqslant 1$

$$\nu(s,1/s^8,\varepsilon) = \|\widehat{f}_>\|_{\frac{1}{s^8},\varepsilon} \leqslant \|\widehat{f}_> \exp\left(\frac{1}{s^8}\frac{\xi^4}{1+\xi^4}\right)\|_2 \lesssim \|\widehat{f}\, e^{s^{-8}}\|_2 \leqslant \|f\|_2 < \infty.$$

We divide the proof in two steps. The main step will be to find s_0 for which $\nu(s_0,1/s_0^8,\varepsilon)$ is bounded for $\varepsilon \in [0,1]$.

First step. Call $M(s)$ the quantity $\frac{e^{s^{-4}}}{\sqrt[3]{s}} + \|\widehat{f}_\sim\|_2$. We rewrite the above inequality (2.3.9) as

$$H(\nu) := (\omega - C\,M(s))\,\nu - C\sum_{j=2}^{5} \nu^j \leqslant C\,M(s).$$

2 This because the exponential weight becomes "lighter" and the support of $\widehat{f}_>$ dwindles.

NOTE ON VARIABLES The function $H(\nu)$ is actually $H(\nu(s,\varepsilon),\omega,s)$ depending implicitly also on s,ε and ω. When f is an extremizer with $\|f\|_2 = 1$, the quantity ω is a fixed number and equals $\mathbf{S}^6$. We will omit it, as well as the dependence of ε.

Consider the function

$$G(\nu) := H(\nu) - \frac{\omega}{2}\nu = \Big(\frac{\omega}{2} - C\,M(s)\Big)\,\nu - C\sum_{j=2}^{5}\nu^j.$$

As we already noticed, $M(s)$ in bounded for any $s > 1$, and

$$M(s) = \frac{e^{s^{-4}}}{\sqrt[3]{s}} + \|\widehat{f_{\sim}}\|_2 \to 0 \qquad \text{as} \quad s \to \infty.$$

For every $\varepsilon > 0$, the function $G(\nu) = G(\nu(s,\varepsilon),s)$ is bounded by

$$G(\nu,s) \leqslant H(\nu,s) \leqslant C\,M(s) < \infty.$$

So $G(\nu,s)$ is bounded by a function of s that goes to 0 as s goes to infinity.

In particular $G(\nu)$ is bounded. Moreover G is strictly concave in ν on $\mathbb{R}_+$ and

$$G(0) = 0 \quad \text{and} \quad \lim_{\nu\to\infty} G(\nu) = -\infty.$$

Thus there exists a point $\nu_{max} \in \mathbb{R}_+$ in which G reaches its maximum.
Let $\nu_0 < \nu_{max}$. For example one can take $\nu_0 = \nu_{max}/2$. Because of the strict concavity, there exists another point ν_1 such that $G(\nu_0) = G(\nu_1)$.
Thus we have
$G^{-1}([0,G(\nu_0)]) = [0,\nu_0] \cup [\nu_1,\infty)$.

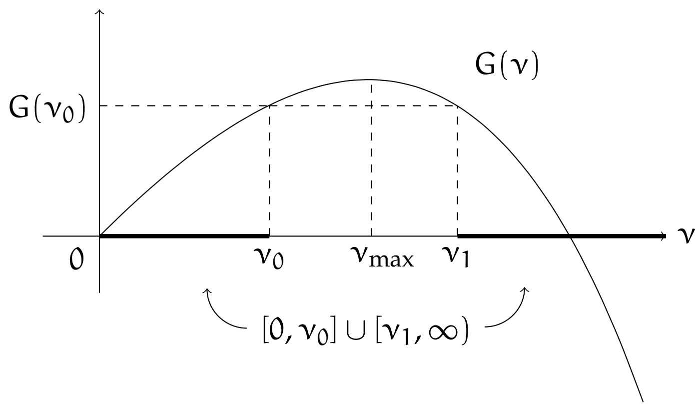


Figure 8.: Graph of $G(\nu)$ and the trapping region $G^{-1}\big([0,G(\nu_0)]\big)$.

We can choose s large enough such that

1. $C\,M(s) < \min\left\{\frac{\omega}{2}, G(\nu_0)\right\}$
2. $\nu(s,1) < \nu_0$.

Fix $s = s_0$ for which the above conditions hold. For this choice of s we have

$$G(\nu,s_0) \leqslant H(\nu,s_0) \leqslant G(\nu_0).$$

In particular, for $\mu_0 = 1/s_0^8$, the required conditions imply

1. $\Rightarrow\ G(\|\widehat{f}_>\|_{\mu_0,\varepsilon}) \leqslant G(\nu_0) \qquad \text{independently of } \varepsilon > 0\,,$

2. $\Rightarrow\ \|\widehat{f}_>\|_{\mu_0,1} \in [0,\nu_0]\ .$

Consider now the function Υ:

$$\varepsilon \mapsto \|\widehat{f}_>\|_{\mu_0,\varepsilon}.$$

This is continuous, so it has to map connected sets into connected sets. Because of (1), $\Upsilon([0,1]) \subset G^{-1}([0,G(\nu_0)]) = [0,\nu_0] \cup [\nu_1,\infty)$. Since $\Upsilon([0,1])$ is connected, it must be contained only in one of the two components. But $\Upsilon(1) \in [0,\nu_0]$, because of (2), so $\|\widehat{f}_>\|_{\mu_0,\varepsilon} \in [0,\nu_0]$ for every positive ε.

Then, by monotone convergence

$$\|\widehat{f}_>\|_{\mu_0,0} = \lim_{\varepsilon\to 0}\|\widehat{f}_>\|_{\mu_0,\varepsilon} = \sup_{\varepsilon>0}\|\widehat{f}_>\|_{\mu_0,\varepsilon} \leqslant \nu_0 < \infty.$$

This shows that the L^2-norm of the tails[3] of $e^{s_0^{-8}|\xi|^4}\widehat{f}$ is finite.

Second step. To obtain the bound on $\widehat{f}$, notice that

$$\|e^{s_0^{-8}|\xi|^4}\widehat{f}\|_2 \leqslant \|e^{s_0^{-8}|\xi|^4}\widehat{f}_<\|_2 + \|e^{s_0^{-8}|\xi|^4}\widehat{f}_>\|_2.$$

The function $e^{s_0^{-8}\xi^4}$ is bounded on $[-s_0^2, s_0^2]$, so

$$\|e^{s_0^{-8}|\xi|^4}\widehat{f}_<\|_2 \leqslant \sup_{[-s_0^2,s_0^2]} e^{s_0^{-8}|\xi|^4}\ \|\widehat{f}\|_2 \lesssim \|\widehat{f}\|_2$$

thus the function $\xi \mapsto e^{s_0^{-8}|\xi|^4}\widehat{f}(\xi)$ belongs to $L^2(\mathbb{R})$. Apply Lemma 2.3.13 to conclude. □

We prove the regularity of extremizers.

Proof of Theorem 2.3.2. We write f using the inverse Fourier transform:

$$f(x) = \int_{\mathbb{R}} e^{ix\xi}\widehat{f}(\xi)\, đ\xi.$$

Then we have

$$\partial f(x) = \partial_x \int_{\mathbb{R}} e^{ix\xi}\widehat{f}(\xi)\, đ\xi = \lim_{h\to 0}\int_{\mathbb{R}} \frac{e^{i(x+h)\xi} - e^{ix\xi}}{h}\widehat{f}(\xi)\, đ\xi.$$

We can use the Dominated convergence theorem to move the limit inside the integral. To see this, denote the integrand by $g_h(\xi)$ and let $g = i\xi e^{ix\xi}\widehat{f}(\xi)$. Since

3 we mean that value on the set $\{\xi \in \mathbb{R} : |\xi| > s_0^2\}$

1. $\lim_{h\to 0} g_h(\xi) = g(\xi)$ pointwise almost everywhere;

2. $|g_h(\xi)| \leqslant \left|\frac{e^{ih\xi}-1}{h}\right| |\widehat{f}(\xi)| \leqslant (1+|\xi|)|\widehat{f}(\xi)| \lesssim e^{\mu|\xi|}\widehat{f}(\xi) \in L^1(\mathbb{R})$,

by applying the Dominated convergence theorem we get

$$\lim_{h\to 0} \int g_h = \int \lim_{h\to 0} g_h = \int g$$

thus

$$\partial f(x) = \int_{\mathbb{R}} \partial_x e^{ix\xi}\widehat{f}(\xi)\, đ\xi = \int_{\mathbb{R}} i\xi\, e^{ix\xi}\widehat{f}(\xi)\, đ\xi.$$

The bound on $\partial^n f(x)$ follows from the one on $(1+|\xi|^n)\widehat{f}(\xi)$, in fact

$$\|\partial^n f\|_\infty \leqslant \left\| \int_{\mathbb{R}} (1+|\xi|^n)\, e^{ix\xi}\widehat{f}(\xi)\, d\xi \right\|_\infty \lesssim \int_{\mathbb{R}} e^{\mu|\xi|}\, |\widehat{f}(\xi)|\, d\xi = \|e^{\mu|\cdot|}\, \widehat{f}\,\|_{L^1(\mathbb{R})}$$

that is finite by Lemma 2.3.13. □

A | FOURIER TRANSFORM

For the convenience of the reader, in this appendix we include some well known results about the Fourier transform. The interested reader can find the details in [SS11a].

RIEMANN-LEBESGUE The Fourier transform of a L^1 function is continuous and decays at infinity.

$$\mathcal{F}\colon L^1(\mathbb{R}^d) \to C_0(\mathbb{R}^d) = \left\{ f\colon \mathbb{R}^d \to \mathbb{R} \text{ continuous, such that } \lim_{|\xi|\to\infty} \widehat{f}(\xi) = 0 \right\}.$$

PLANCHEREL The Fourier transform $\mathcal{F}$ is an isometry on L^2. More precisely

$$\mathcal{F}\colon L^2(\mathbb{R}^d, dx) \to L^2\left(\mathbb{R}^d, \frac{d\xi}{(2\pi)^{\frac{d}{2}}}\right).$$

One has

$$\|\cdot\|_{L^2(\mathbb{R}^d)} = (2\pi)^{-\frac{d}{2}} \|\mathcal{F}[\,\cdot\,]\|_{L^2(\mathbb{R}^d)}. \tag{A.0.1}$$

CONVOLUTION On $\mathbb{R}^d$ the following identities hold

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g} \qquad \widehat{f \cdot g} = \frac{1}{(2\pi)^d} \widehat{f} * \widehat{g}, \tag{A.0.2}$$

$$\widecheck{f \cdot g} = \check{f} * \check{g} \qquad \widecheck{f * g} = (2\pi)^d \check{f} \cdot \check{g}. \tag{A.0.3}$$

INVERSION

Proposition A.0.1. *Let* $f, \widehat{f} \in L^1(\mathbb{R}^d)$. *Then*

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi. \tag{A.0.4}$$

Fourier transform of a measure

It is possible to define the Fourier transform of a *finite* measure. The following is adapted from [Tar07, page 74].

Let be μ a finite measure on $\mathbb{R}^d$. Consider the pairing with continuous compactly supported functions $\varphi \in C_c(\mathbb{R}^d)$ given by

$$\langle \mu, \varphi \rangle := \int_{\mathbb{R}^d} \varphi(x)\, d\mu.$$

We have that

$$|\langle \mu, \varphi \rangle| \leqslant C\|\varphi\|_\infty \quad \text{for all } \varphi \in C_c(\mathbb{R}^d).$$

If the above bound holds, the maps $\varphi \mapsto \langle \mu, \varphi \rangle$ extends in an unique way, with the same bound, to the Banach space $C_b(\mathbb{R}^d)$ of continuous bounded functions equipped with the sup norm $\|\cdot\|_\infty$. We can define the Fourier transform of such a measure as

$$\widehat{\mu}(\xi) := \langle \mu, e^{-i\langle \cdot, \xi \rangle} \rangle.$$

By Dominated convergence theorem, $\widehat{\mu}$ is continuous.

Fourier transform on tempered distributions $\mathscr{S}'(\mathbb{R})$

It is possible to apply the Fourier transform to a larger class objects. There are several books on the topic, for example the one by Strichartz [Str03].

Consider a local integrable function f. We indicate with T_f the corresponding distribution in $\mathscr{S}'(\mathbb{R})$. This is the tempered distribution such that, for every $\varphi \in \mathscr{S}(\mathbb{R})$, it is defined as

$$\langle T_f, \varphi \rangle := \int_{\mathbb{R}} f(x)\varphi(x)\, dx.$$

Let $a \in \mathbb{R}$, and $\delta_a \in \mathscr{S}'(\mathbb{R})$. Then, for every $\varphi \in \mathscr{S}(\mathbb{R})$ we have

$$\langle \widehat{\delta_a}, \varphi \rangle = \langle \delta_a, \widehat{\varphi} \rangle = \widehat{\varphi}(a) = \int e^{-i\langle a, x \rangle} \varphi(x)\, dx = \langle T_{e^{-i\langle a, \cdot \rangle}}, \varphi \rangle$$

so

$$\widehat{\delta_a} = e^{-i\langle a, \cdot \rangle} \tag{A.0.5}$$

in distributional sense, meaning that

$$\int_{\{|x|<R\}} e^{-i\langle \cdot, x \rangle}\, dx \xrightarrow{R \to \infty} 2\pi\, \delta_0$$

in the sense of tempered distributions (i.e. in the weak-$\star$-topology on $\mathscr{S}'(\mathbb{R})$).

LITTLEWOOD–PALEY THEORY

We provide a short introduction to the Littlewood-Paley theory. The interested reader can find an entire chapter on this topic in [Gra08, Chapter 5].

Lemma A.1.1. *There exists a smooth compactly supported function $\psi \in C_c^\infty(\mathbb{R} \setminus \{0\})$, non-negative, radial, such that*

$$\sum_{j \in \mathbb{Z}} \psi(2^{-j}x) = 1, \quad \forall x \in \mathbb{R} \setminus \{0\}$$

and for any $x \neq 0$ the sum consists of at most two terms.

Proof. Let $\chi \in C^\infty(\mathbb{R})$ smooth, radial, non-negative, such that $\chi \equiv 1$ on the ball of radius 1 and $\chi \equiv 0$ on the complement of the ball of radius 2. Let $\psi(x) = \chi(x) - \chi(2x)$. For any $N \in \mathbb{N}$, the sum of the rescaled $\psi(2^{-j}x)$ is the telescopic sum:

$$\sum_{j=-N}^{N} \psi_j(x) = \sum_{j=-N}^{N} \left[\chi(2^{-j}x) - \chi(2^{-j+1}x)\right] = \chi(2^{-N}x) - \chi(2^{N+1}x)$$

and for any $x \neq 0$ there exists $N \in \mathbb{N}$ such that $\chi(2^{-N}x) = 1$ and $\chi(2^{N+1}x) = 0$. □

Definition A.1.2 (Littlewood-Paley projector)**.** Let $\psi_j(x) := \psi(2^{-j}x)$. We define

$$P_j f := (\psi_j \widehat{f})^{\vee}, \quad f \in \mathscr{S}(\mathbb{R}).$$

When $\xi \in \operatorname{supp}(\psi_j)$ then

$$\xi \in A_j, \text{ where } A_j = \left\{\xi : 2^j \leqslant |\xi| < 2^{j+1}\right\}.$$

We indicate the condition $\xi \in A_j$ with $|\xi| \simeq 2^j$. Thus

$$\widehat{P_j f} = \mathbb{1}_{A_j} \widehat{f}.$$

Definition A.1.3 (Littlewood-Paley square function)**.**

$$Sf(x) := \left(\sum_{j \in \mathbb{Z}} |P_j f(x)|^2\right)^{\frac{1}{2}}.$$

We have

$$\|Sf\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \sum_{j \in \mathbb{Z}} |P_j f|^2 = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} |P_j f|^2 = \sum_{j \in \mathbb{Z}} \|P_j f\|_{L^2(\mathbb{R})}^2 \simeq \|f\|_{L^2(\mathbb{R})}^2. \tag{A.1.1}$$

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