A plausibility model for Iterated Eliminating Regret-Dominated Strategies algorithm*†

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Abstract

This paper is based upon Cui, Luo, and Sim (2013) which develops an epistemic model for the Iterated Eliminating Regret-Dominated Strategies (IERS) in Halpern and Pass (2012). In section 1 I will present the IERS and the relevant results by Cui et al. (2013). In section 2 I will highlight a possible mistake in their paper and propose a solution according to the IERS epistemic characterization. Then, in section 3, I will develop a plausibility regret model to avoid the elimination of strategies so that also repeated games can be modeled. In section 4 I will compare this plausibility characterization to a preexisting one. In section 5 I will give an example based on the defined plausibility model of IERS. In section 6 I will draw the conclusions, study the IERS using the $\mu-$calculus and I will sketch some possible future work.

1 Introduction

In this paper we will study the Iterated Eliminating Regret-Dominated Strategies (IERS) algorithm. The purpose of this study is to shift the applicability of this algorithm from a one shot game to a repeated game. We will see the formalization introduced by Cui et al. (2013) and then we will see a proposal for a different characterization of the algorithm which enables to keep all the strategies and create levels of plausibility among them. Then we will compare this plausibility characterization to a preexisting one. We

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will also see an example of an iterated game using the plausibility characterization of IERS; finally we will give a formalization of IERS using the $\mu-$calculus.

Definition 1. A strategic form game with pure strategies is a tuple of $G = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, where:

- N is the set of players in the game G.
- S_i is the set of strategies for player i.
- u_i is a function that assigns a real value to every strategy profile $s = (s_1, .., s_n)$.

We denote by $S_{-i} = S_1 \times .. \times S_{i-1} \times S_{i+1} \times .. \times S_n$ the set of strategy profiles other than i. When we want to focus on the strategy of player i, we denote the strategy profile $s \in S$ as $s = (s_i, s_{-i})$ where $s_i \in S_i$ and $s_{-i} \in S_{-i}$.

Let's now define the regret game of a given strategic form game:

Definition 2. For a normal form game $G = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, its strategic regret game is a 3-tuple of $G' = \langle N, \{S_i\}_{i \in N}, \{re_i\}_{i \in N} \rangle$, where re_i stands for player i's expost regret associated with any profile of pure strategies (s_i, s_{-i}), which is calculated as follows:

$$re_i(s_i, s_{-i}) = \max\{u_i(s'_i, s_{-i}) | \forall s'_i \in S_i\} - u_i(s_i, s_{-i}).$$

Meaning the regret of choosing s_i for player i when his opponents choose s_{-i}.

Now we define a way a player i can confront two strategies on the basis of their maximal regret.

Definition 3. Given a strategic regret game $G' = \langle N, \{S_i\}_{i \in N}, \{re_i\}_{i \in N} \rangle$, let s_i and s'_i be two strategies for player i. We say that s_i is regret dominated by s'_i if $Re_i(s'_i) < Re_i(s_i)$. Where

$$Re_i(s_i) = \max\{re_i(s_i, s_{-i}) | \forall s_{-i} \in S_{-i}\}.$$

We say that a regret dominated strategy s_i is regrettable for player i. And for $S' \subseteq S$ we say that strategy $s'_i \in S_i$ is unregretted with respect to S'_i if no strategy in S'_i regret dominates s'_i on S'_{-i}.

Let's now see the recursive elimination process of IERS:

Definition 4. Given a strategic regret game $G' = \langle N, \{S_i\}_{i \in N}, \{re_i\}_{i \in N} \rangle$, let IUD be the set of iterated regret-undominated strategies of G' recursively defined by:

$$IUD = \prod_{i \in N} IUD_i.$$

Where $IUD_i = \bigcap_{m \geq 0} IUD_i^m$ with $IUD_i^0 = S_i$ and

$$RD_i^0 = \{s_i | s_i \in IUD_i^0 \text{ is regrettable with respect to } IUD_i^0 \text{ in } G'\}.$$

And for $m \geq 1$ we have the following: $IUD_i^m = IUD_i^{m-1} \setminus RD_i^{m-1}$ and $RD_i^{m-1} = \{s_i | s_i \in IUD_i^{m-1}$ is regrettable with respect to IUD_i^{m-1} in $G'\}$.

Let's now give an example to understand the previous definitions.
In Table 1 we have: $N = \{player1, player2\}$, $S_1 = \{A, B, C\}$, $S_2 = \{a, b, c\}$, $u(A, b) = (1, 2)$ (and so on for all the other outcomes).
The computed regret values are in Table 2. For example, $re_1(A, a) = 3 - 0 = 3$.
Let's see how the IERS works: $Re_1(A) = 4$, $Re_1(B) = 2$, $Re_1(C) = 2$, so strategy A is dominated. We will then cancel the first row from the table (because the corresponding strategy is an element of RD_1^0). So, $IUD_1^1 = \{B, C\}$ and similarly $RD_2^0 = \{b\}$ and $IUD_2^1 = \{a, c\}$ (so we also cancel the second column from the table). At the second iteration we find that $IUD_1^1 = IUD_1^2 = \{B, C\}$ (i.e. player 1 has no dominated strategy), $RD_2^2 = \{c\}$ and $IUD_2^2 = \{a\}$ (so we cancel the third column from the table). At the third iteration we find that $IUD_1^3 = \{C\}$ and $RD_1^3 = \{B\}$ while $IUD_2^3 = \{a\}$ (so we cancel the second row from the table). In the end we find that $IUD = \{(C, a)\}$.

player 1 - player 2	a	b	c
A	(0,0)	(1,2)	(0,0)
B	(1,3)	(0,0)	(4,3)
C	(3,4)	(2,0)	(2,3)

Table 1: A two player game.

player 1 - player 2	a	b	c
A	(3,2)	(1,0)	(4,2)
B	(2,0)	(2,3)	(0,0)
C	(0,0)	(0,4)	(2,1)

Table 2: Regret game of the game represented in Table 1.

Theorem 5. *Let $G' = \langle N, \{S_i\}_{i \in N}, \{re_i\}_{i \in N} \rangle$ be a strategic regret game. If S is a closed, non-empty set of strategies, then IUD is non-empty.*

Cui et al. (2013) manage to represent the IERS through the logic PAL of public announcement. Let's see how they do it:

Definition 6. Given a regret game G', a regret game logic (G'-logic) is a logic that contains atomic propositions in the following form:

- Pure strategy symbols s_i, s_i'... so the interpretation for s_i is that player i chooses strategy s_i.
- The symbol Ra_i^{re} means that player i is rational. Br_i^* means the best response of player i. GS means the game solution of the game through algorithm IERS.
- Atomic propositions in the form $s_i \succcurlyeq s_i'$ mean that strategy s_i is weakly regret dominant over strategy s_i' for player i while $s_i \succ s_i'$ means that strategy s_i is better than strategy s_i' for player i.

Definition 7. Given a regret game G', a frame of G'-logic is $\mathcal{F}' = \langle W, \{\sim_i\}_{i\in N}, \{f_i\}_{i\in N}\rangle$, where

- W ($\neq \emptyset$) consist of all players' pure strategy profiles (i.e. in the previous example of Table 1 $(A, a) \in W$).
- $\sim_i$ is an epistemic accessibility relation for player i, which is defined as the equivalence relation of agreement of profiles in the i'th coordinate (i.e. in the previous example of Table 1 $(A, a) \sim_1 (A, b) \sim_1 (A, c)$).
- f_i is a pure strategic function, which satisfies the following property: if $w \sim_i v$ then $f_i(w) = f_i(v)$ (i.e. in the previous example of Table 1 if $w = (A, a)$ and $v = (A, b)$ then $f_1(w) = A = f_1(v)$).

Observation 8. We will also use the followings:

- $R_i(w) = \{v|w \sim_i v \text{ where } w, v \in W\}$, the set of worlds that i believes possible from world w.
- $\|s_i\| = \{w \in W|f_i(w) = s_i\}$ is the set of worlds where the player i chooses the strategy s_i.

Let's finally see the definition of a model:

Definition 9. An epistemic game model $M_{G'}$ over G'-logic is obtained by incorporating the following valuation on $\mathcal{F}'$:

- $M_{G'}, w \vDash s_i \iff w \in \|s_i\|$.
- $M_{G'}, w \vDash s_i \succcurlyeq s_i' \iff \exists v \in \|s_i'\|$ such that $re_i(s_i, f_{-i}(w)) \leq re_i(s_i', f_{-i}(v))$.
- $M_{G'}, w \vDash s_i \succ s_i' \iff \forall v \in \|s_i'\|$ we have that $re_i(s_i, f_{-i}(w)) < re_i(s_i', f_{-i}(v))$.
- $M_{G'}, w \vDash Ra_i^{re} \iff M_{G'}, w \vDash \bigwedge_{\{a\neq f_i(w)\}} K_i(f_i(w) \succcurlyeq a)$.

Observation 10. Let's see an example of the last point of the previous definition using Table 1 and 2. We can see that $A \succeq B$ holds in world (A,b). This is because there is world (B,a) such that $1 = re_1(A, f_2(A, b)) \leq re_1(B, f_2(B, a)) = 2$. But Ra_1^{re} fails at the world (A,b) because there is world $(A, a) \in R_1(A, b)$ such that $M_{G'}, (A, a) \nvDash A \not\succeq B$. Similarly, the worlds (A, a) and (A, c) do not satisfy Ra_1^{re}.

Let's see briefly some results developed in Cui et al. (2013):

Theorem 11. *Every finite general epistemic regret-game model has worlds in which Ra^{re} is true, where $Ra^{re} = \bigwedge_{i \in N} Ra_i^{re}$.*

Theorem 12. *Rationality is epistemically introspective. The formula*

$$Ra_i^{re} \to K_i Ra_i^{re}$$

is valid in a general epistemic regret-game model.

Observation 13. At this point the authors states that, thanks to the two previous results, it is possible to successively remove the worlds where Ra^{re} doesn't hold. They obtain this by iteratively announcing rationality.

Theorem 14. *Given a full epistemic model $M_{G'}$ based on finite strategic-form game G' with regret, we define a general epistemic game model $M^*_{G'}$ as the fix-point model obtained after iterated announcement of rationality over $M_{G'}$. $M^*_{G'}$, where arbitrary world w is, is stable by repeated announcements of Ra^{re} in $M_{G'}$ for all the players of game G' if and only if $f(w) \in IUD$, i.e.*

$$w \in \sharp(Ra^{re}, M_{G'}) \leftrightarrow f(w) \in IUD.$$

2 Traveler's Dilemma

In the paper Cui et al. (2013) the authors propose a solution for the Traveler's Dilemma, let's introduce the dilemma and see how their solution could be wrong.
Two friends take a plane to attend a surf competition. The airplane company looses their two suitcases, that contain exactly the same swimming suits. They are asked to give a value to their luggage from 2 to 100 without the possibility of consulting each other. The company will refund both of them the lower value and take 2 from the one who gave the higher value and give it to the other. In Table 3 it is possible to find the outcomes of the strategies. The Nash equilibrium is given by choosing the strategies (2,2), which give the lowest outcome; this is a solution far from the strategies (97,97) which represent the experimental results.

player 1 - player 2	100	99	98	...	3	2
100	(100,100)	(97,101)	(96,100)	...	(1,5)	(0,4)
99	(101,97)	(99,99)	(96,100)	...	(1,5)	(0,4)
98	(100,96)	(100,96)	(98,98)	...	(1,5)	(0,4)
⋮	⋮	⋮	⋮	⋱	⋮	⋮
3	(5,1)	(5,1)	(5,1)	...	(3,3)	(0,4)
2	(4,0)	(4,0)	(4,0)	...	(4,0)	(2,2)

Table 3: Table representing the game of the Traveler's Dilemma.

In Cui et al. (2013) paper the authors claim that after public announcing Ra^{re} one time it can be attained the following regret sub-model.

player 1 - player 2	100	99	98	97	96
100	(1,1)	(3,0)	(3,1)	(3,2)	(2,3)
99	(0,3)	(1,1)	(3,0)	(3,1)	(2,2)
98	(1,3)	(0,3)	(1,1)	(3,0)	(2,1)
97	(2,3)	(1,3)	(0,3)	(1,1)	(2,0)
96	(3,2)	(2,2)	(1,2)	(0,2)	(0,0)

Table 4: Table introduced by Cui et al. (2013) representing the regret game of the Traveler's Dilemma.

Thus after public announcing another time Ra^{re} they find (97,97) as the only game solution.
It seems that this solution is wrong. By computing the whole regret game as in Table 5, it is easy to see that we can find the regret sub-model $[100, 96] \times [100, 96]$ after public announcing 94 times Ra^{re}. This is also the game solution. In Cui et al. (2013) is obtained the different result (97,97) because the authors compute the regret game on the Traveler's Dilemma game $[100, 96] \times [100, 96]$. Thereby they obtain a different regret game. But this is different from their original definition of the epistemic IERS. Although their definition of epistemic IERS is interesting, it doesn't work for the purpose for which it was created: to find that the game solution is (97,97). Correctly applying their definition of epistemic IERS, one finds out that the game solution is $[100, 96] \times [100, 96]$, which is still a good solution. It is certainly better than (2,2).

player 1 - player 2	100	99	98	97	96	95	...	3	2
100	(1,1)	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	...	(3,96)	(2,97)
99	(0,3)	(1,1)	(3,0)	(3,1)	(3,2)	(3,3)	...	(3,95)	(2,96)
98	(1,3)	(0,3)	(1,1)	(3,0)	(3,1)	(3,2)	...	(3,94)	(2,95)
97	(2,3)	(1,3)	(0,3)	(1,1)	(3,0)	(3,1)	...	(3,93)	(2,94)
96	(3,3)	(2,3)	(1,3)	(0,3)	(1,1)	(3,0)	...	(3,92)	(2,93)
95	(4,3)	(3,3)	(2,3)	(1,3)	(0,3)	(1,1)	...	(3,91)	(2,92)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
3	(96,3)	(95,3)	(94,3)	(93,3)	(92,3)	(91,3)	...	(1,1)	(2,0)
2	(97,2)	(96,2)	(95,2)	(94,2)	(93,2)	(92,2)	...	(0,2)	(0,0)

Table 5: Table representing the regret game of the Traveler's Dilemma.

3 A plausibility model for IERS

I think that it is restricting to use the update to represent the IERS algorithm. I will therefore develop a different model where radical upgrades are used instead of updates. This could give the players the chance to change strategy across time in case they change belief (for example through a radical upgrade caused by sources like friends, mentors, books, papers, et cetera).

Definition 15. A multi-agent plausibility model is an $\mathbf{S} = \langle S, \leq_a, \sim_a, \| \cdot \|, s^* \rangle_{a \in N}$, where:

- S is a set o possible worlds.
- N is a finite set of agents.
- $\leq_a$ is a preorder (i.e. reflexive and transitive) on S: a's plausibility relation.
- $\sim_a$ is an equivalence relation on S: a's epistemic possibility.
- $\| \cdot \| : \Phi \to \wp(S)$ is a valuation map for a set Φ.
- a designated state (the actual world s^*).

This model goes with the following three conditions:

1) Plausibility implies possibility: $s \leq_a t$ implies $s \sim_a t$.
2) Indistinguishable states are comparable: $s \sim_a t$ implies $s \leq_a t$ or $s \geq_a t$.
3) The preorders $\leq_a$ are converse well-founded: no infinite ascending chains $s_0 \leq_a s_1 \leq_a \dots$.

Let's now build our regret game logic for plausibility models.

Definition 16. Given a game G and its regret game G', a plausibility regret game logic (G'_p-logic) is a logic that contains atomic proposition in the following form:

- Pure strategy symbols $s_i, s'_i \ldots$ .The interpretation for s_i is that player i chooses strategy s_i.
- The symbol Ra_i^{re} indicates player i is rational. Br_i^* means the best response of player i. GS means the game solution of the game through algorithm IERS.
- Atomic propositions in the form $s_i \succcurlyeq s'_i$ mean that strategy s_i is weakly regret dominant over strategy s'_i for player i while $s_i \succ s'_i$ means that strategy s_i is better than strategy s'_i for player i.

We also ask G'_p to contain the symbols $\{B_a, K_a\}_{a \in N}$ representing the Belief and the knowledge of agent a, respectively.

Definition 17. Given a game G', a plausibility frame of G'_p-logic is $\mathcal{F}' = \langle W, \{\leq_i\}_{i \in N}, \{\sim_i\}_{i \in N}, s^*, \{f_i\}_{i \in N} \rangle$, where

- W ($\neq \emptyset$) consist of all players' pure strategies profile.
- $\sim_i$ is an equivalence relation on S: i's epistemic possibility.
- $\leq_i$ is a preorder on S: i's plausibility relation.
- f_i is a pure strategic function, which satisfies the following property: if $w \sim_i v$ then $f_i(w) = f_i(v)$.
- a designated state (the actual world) s^*.

This frame goes with the following four conditions:

1) Plausibility implies possibility: $s \leq_a t$ implies $s \sim_a t$.
2) Indistinguishable states are comparable: $s \sim_a t$ implies $s \leq_a t$ or $s \geq_a t$.
3) The preorders $\leq_a$ are conversely well-founded: no infinite ascending chains $s_0 \leq_a s_1 \leq_a \ldots$.
4) All the worlds for all the agents are possible (i.e. for every player i and every two worlds $v, w \in W$ we have that $v \sim_i w$).

Observation 18. I add one more condition to the plausibility frame for regret games respectively to the standard definition for plausibility model. According to condition 4 in the previous definition, all the worlds are equally possible. By doing radical upgrades of rationality for each player we will find which worlds are more plausible, until we reach a fix-point. I will show this below.

Observation 19. As before we will use the followings:

- $R_i(w) = \{v | w \sim_i v \text{ and } f_i(w) = f_i(v) \text{ where } v \in W\}$, the set of worlds that are possible from world w for player i which have the same i-th strategy as w.
- $\|s_i\| = \{w \in W | f_i(w) = s_i\}$ is the set of worlds where the player i chooses the strategy s_i.

Let's see the definition of our model.

Definition 20. A plausibility game model $M_{G'_p}$ over G'_p-logic is obtained by incorporating the following valuation on the plausibility frame $\mathcal{F}'$:

- $M_{G'_p}, w \vDash s_i \iff w \in \|s_i\|$.
- $M_{G'_p}, w \vDash s_i \succcurlyeq s'_i \iff \exists v \in \|s'_i\|$ such that $re_i(s_i, f_{-i}(w)) \leq re_i(s'_i, f_{-i}(v))$.
- $M_{G'_p}, w \vDash s_i \succ s'_i \iff$
  $\forall v \in \|s'_i\|$ we have that $re_i(s_i, f_{-i}(w)) < re_i(s'_i, f_{-i}(v))$.
- $M_{G'_p}, w \vDash Ra_i^{re} \iff (w \in Max_{\leq_i}(W)$ and
  $\forall v \in R_i(w) \cap \bigcap_{i \in N} Max_{\leq_i}(W)$, $M_{G'_p}, v \vDash \bigwedge_{\{a \neq f_i(w)\}}(f_i(w) \succcurlyeq a))$.
- $M_{G'_p}, w \vDash K_i\phi \iff M_{G'_p}, v \vDash \phi$ for all $v \in R_i(w)$.
- $M_{G'_p}, w \vDash B_i\phi \iff M_{G'_p}, v \vDash \phi$ for all $v \in Max_{\leq_i}(R_i(w))$.

Where $Max_{\leq_i}(P) = \{s \in P \mid t \leq_i s \ \forall t \in P\}$.

Theorem 21. *Every finite plausibility regret-game model has worlds in which Ra^{re} is true, where $Ra^{re} = \bigwedge_{i \in N} Ra_i^{re}$.*

Proof. The proof is the same as the one given for Theorem 11 by Cui et al. (2013). □

Theorem 22. *Given a plausibility regret-game model $M^*_{G'_p}$,which is the positive fix-point of iterated radical upgrade of rationality, rationality is doxastically introspective. The following formula is valid*

$$Ra_i^{re} \to B_i Ra_i^{re}.$$

Proof. Consider a plausibility regret-game model $M^*_{G'_p}$ and an arbitrary world w in it such that $M^*_{G'_p}, w \vDash Ra_i^{re}$ but $M^*_{G'_p}, w \nvDash B_i Ra_i^{re}$. Then there is a $v \in Max_{\leq_i}(W)$ such that $M^*_{G'_p}, v \nvDash Ra_i^{re}$. By Definition 20, $f_i(v)$ is a regret dominated strategy for player i by some of his strategies. Since $v \in Max_{\leq_i}(W)$ and the model is a positive fix point of radical upgrade of Ra^{re} then v is not a dominated strategy for i. □

Theorem 23. *Given a plausibility regret-game model $M_{G'_p}$ based on a finite strategic-form game G' with regret, we define $M^*_{G'_p}$ as the fix-point model obtained after iterated radical upgrade of rationality over $M_{G'_p}$. Given an arbitrary world w, w is among the maximal worlds for all the players of game G' in the model $M^*_{G'_p}$ if and only if $f(w) \in IUD$, i.e.*

$$w \in \sharp(Ra^{re}, \bigcap_{i \in N} Max_{\leq_i}(W)) \leftrightarrow f(w) \in IUD.$$

Proof. ($\rightarrow$) If $w \in \sharp(Ra^{re}, \bigcap_{i\in N} Max_{\leq_i}(W))$, then $M^*_{G'_p}, w \models Ra^{re}$, i.e. $M^*_{G'_p}, w \models \bigwedge_{i\in N} Ra_i^{re}$. First we show: $\forall i \in N$, $f_i(w) \notin RD_i^0$. Suppose not. Then $\exists i \in N$ such that $f_i(w) \in RD_i^0$, that is, $f_i(w)$ of player i is regret-dominated in G' by some other strategy $s'_i \in S_i = IUD_i^0$. It means $Re_i(f_i(w)) > Re_i(s'_i)$, thus by definition of $Re_i(\cdot)$, we have

$$\max\{re_i(f_i(w), s_{-i}) | \forall s_{-i} \in S_{-i}\} > \max\{re_i(s'_i, s_{-i}) | \forall s_{-i} \in S_{-i}\}.$$

Now set some $s'_{-i} \in S_{-i}$ satisfying $re_i(f_i(w), s'_{-i}) = Re_i(f_i(w))$, and set $s''_{-i} \in S_{-i}$ satisfying $re_i(s'_i, s''_{-i}) = Re_i(s'_i)$. Thus, by the previous inequality we have

$$re_i(f_i(w), s'_{-i}) > re_i(s'_i, s''_{-i}).$$

Furthermore, set $v' \in R_i(w) \cap \|s'_{-i}\|$. Then by the previous inequality

$$re_i(f_i(w), f_i(v')) > re_i(s'_i, s''_{-i}).$$

Thus, considering $\forall v \in \|s'_i\|$, $re_i(s'_i, s''_{-i}) \geq re_i(s'_i, f_{-i}(v))$, then we can then find from the previous inequality

$$\forall v \in \|s'_i\|,\ re_i(f_i(w), f_{-i}(v')) > re_i(s'_i, f_{-i}(v)).$$

According to $f_i(w) = f_i(v')$, which follows from $v' \in R_i(w)$ and the definition of the plausibility frame, we find

$$\forall v \in \|s'_i\|,\ re_i(f_i(v'), f_{-i}(v')) > re_i(s'_i, f_{-i}(v)).$$

Then we have that $M_{G'_p}, v' \nvDash f_i(v') \succcurlyeq s'_i$. From $f_i(w) = f_i(v')$ we have that $M_{G'_p}, v' \nvDash f_i(w) \succcurlyeq s'_i$, since $v' \in R_i(w) \cap \bigcap_{i\in N} Max_{\leq_i}(W)$. Then, by definition, we obtain that $M_{G'_p}, w \nvDash Ra_i^{re}$. This is against our hypothesis. Since $\forall w \in W$ $f_i(w) \in IUD_i^0$, it follows that $f_i(w) \in IUD_i^0 \setminus RD_i^0 = IUD_i^1$.

Let's see now the inductive step. For a given integer $m \geq 1$, suppose that $\forall j \in N$, $f_j(w) \in IUD_j^m$, then we need to show that $f_j(w) \notin RD_j^m$. Suppose not. Then there is player i such that $f_i(w) \in RD_i^m$. That is, $f_i(w)$ is a regret dominated strategy in G'^m by some other strategy $s'_i \in IUD_i^m$. Then we have

$$\max\{re_i(f_i(w), s_{-i})|\forall s_{-i} \in IUD^m_{-i}\} > \max\{re_i(s'_i, s_{-i})|\forall s_{-i} \in IUD^m_{-i}\}.$$

By the induction hypothesis, $\forall j \in N$, $f_j(w) \in IUD^m_j$. Thus we have that

$$\max\{re_i(f_i(w), f_{-i}(v))|\forall v \in R_i(w) \text{ such that } f_i(v) \in IUD^m_i\} >$$
$$\max\{re_i(f_i(w'), f_{-i}(v))|\forall v \in R_i(w) \text{ such that } f_i(v) \in IUD^m_i\}.$$

Where $w' \in \|s'_i\|$. Similar to the above proof, we can conclude that $M_{G'_p}, w \nvDash Ra^{re}_i$. This is in contradiction with the hypothesis that $M_{G'_p}, w \vDash Ra^{re}_i$. So $f_i(w) \in IUD^m_i \setminus RD^m_i = IUD^{m+1}_i$. Then for induction we have that $\forall i \in N$, $f_i(w) \in IUD_i$.

($\leftarrow$) Let $f(w) \in IUD = \bigcap_{m \geq 0} IUD^m$. Then $\forall i \in N$ $f_i(w)$ is never regret dominated in IUD^m. It means that after m radical upgrades of Ra^{re}, $M_{G'^m}, w \vDash Ra^{re}$, where $M_{G'^m}$ is the plausibility model related to submodel G'^m. Due to the arbitrary m we have that $w \in \sharp(Ra^{re}, \bigcap_{i \in N} Max_{\leq i}(W))$.

□

4 Comparison to a preexisting plausibility formalization for IERS

After writing the previous section I found that in Cui (2012) the author already drew a plausibility formalization for IERS. Let's see the given semantic definition.

Definition 24. An epistemic model $M_{G'}$ over G'-logic is obtained by incorporating the following valuation on a $F'_{G'}$.

- $M_{G'}, w \vDash s_i \iff w \in \|s_i\|$.
- $M_{G'}, w \vDash s_i \succcurlyeq s'_i \iff \exists v \in \|s'_i\|$ such that $re_i(s_i, f_{-i}(w)) \leq re_i(s'_i, f_{-i}(v))$.
- $M_{G'}, w \vDash s_i \succ s'_i \iff \forall v \in \|s'_i\|$ we have that $re_i(s_i, f_{-i}(w)) < re_i(s'_i, f_{-i}(v))$.
- $M_{G'}, w \vDash Ra^{re'}_i \iff M_{G'}, w \vDash \bigwedge_{\{a \neq s_i\}} B_i(s_i \succcurlyeq a)$.[1]

According to Cui (2012), if we give a plausibility model to the regret game showed in Table 2, by iterative radical upgrade of rationality ($\Uparrow Ra^{re}$) we find exactly the same result as in the public announcement case. But this

[1]The original definition in Cui (2012) is $M_{G'}, w \vDash Ra^{re'}_i \iff M_{G'}, w \vDash s_i \wedge (\bigwedge_{\{a \neq s_i\}} B_i(s_i \succcurlyeq a))$, but it seems to be redundant to ask a world to satisfy its own strategy ($f_i(w) = s_i$). Thus I hide "$s_i \wedge$" to make the formula clearer.

seems not to be the case. Public announcement deletes the worlds that do not satisfy the announced property. On the other hand, radical upgrade doesn't delete worlds; radical upgrade changes the preorder putting on top those worlds satisfying the upgraded property. The previous definition doesn't take into account this fact. After the first radical upgrade of rationality any other radical upgrade of rationality will not change the model because the regret outcomes are compared again among all worlds. But this problem can be overtaken by analyzing only the worlds that are already in the maximal set of the preorder. Thus to start studying the rationality of a world, such a world should be already among the maximal worlds of the preorder. This fact is taken into account in Definition 20.

5 Benedick and Beatrice

I will use an example inspired by William Shakespeare through the plausibility regret game model defined in section 3. Benedick from Padua (Ben) and Beatrice (Bea), love each other but one doesn't know about the other person's feelings. Both of them are also very proud, so they interpret the other person's mocking attitude as the absence of love. We assume that $\neg BeaLBen \leftrightarrow BeaMBen$ and similarly for Benedick, where "BeaLBen" means that Beatrice loves Benedick and "BeaMBen" stands for Beatrice mocks Benedick. This situation can be modeled in the *belief plausibility model* of Figure 1 (where arrows represent the plausibility relations).

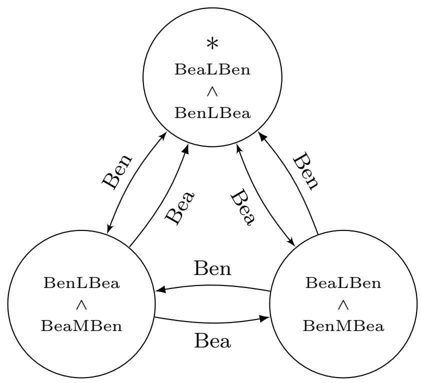

Figure 1: Figure representing the plausibility belief model.

Each one of them can choose between two equally plausible strategies: declaring their love or mocking the other person. Due to their pride they will also "feel good" only for mocking the other person (even better if in such a case the other person declares his/her love!). The outcomes of this game can be found in Table 6.

Ben - Bea	BeaLBen	BeaMBen
BenLBea	(2,2)	(0,2)
BenMBea	(2,0)	(1,1)

Table 6: Table representing the strategy game of Benedick and Beatrice.

Both Beatrice and Benedick can't decide which action to choose so they think the possible outcomes and understand that there are two Nash equilibria: (BenLBea, BeaLBen) and (BenMBea, BeaMBen). So, each one of them may compute the regret game (Table 7) and compute the IERS (Figure 2). Applying the IERS they both find that the best solution is given by choosing the Mock strategy: (BenMBea, BeaMBen).

Ben - Bea	BeaLBen	BeaMBen
BenLBea	(0,0)	(1,0)
BenMBea	(0,1)	(0,0)

Table 7: Table representing the regret strategy game of Benedick and Beatrice.

$$\begin{tikzcd} (0,0)^* \arrow[r, "Beatrice"] \arrow[d, "Benedick"'] & (1,0) \arrow[d, "Benedick"] \\ (0,1) \arrow[r, "Beatrice"] & (0,0) \end{tikzcd}$$

Figure 2: Figure representing the regret strategy game of Benedick and Beatrice after the IERS upgrade.

After being mocked both of them will upgrade their belief plausibility model by applying the following radical upgrades: $\Uparrow_{Ben} \neg BeaLBen$ and $\Uparrow_{Bea} \neg BenLBea$. (They both apply a radical upgrade, and not an update, because at the bottom of their hearts they still hope that the other person loves him/her). Thus the plausibility belief model will be changed as the one in Figure 3.

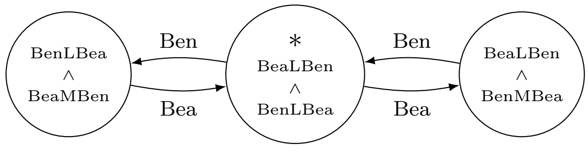

Figure 3: The plausibility belief model after mocking.

Since $B_{Bea}BenMBea$ and $B_{Ben}BeaMBen$, these two beliefs will induce the following two upgrades in the plausibility strategy model:

$\Uparrow_{Ben} \neg BenLBea$ and $\Uparrow_{Bea} \neg BeaLBen$. But, these two upgrades leave the plausibility strategy model unchanged. Thus we have found a fix-point between the two models! After some time their friends manage to speak to each of them and convince them that the other loves him/her. So they apply the following radical upgrades in their plausibility belief model: $\Uparrow_{Ben} BeaLBen$ and $\Uparrow_{Bea} BenLBea$ which will give the plausibility belief model as in Figure 4.

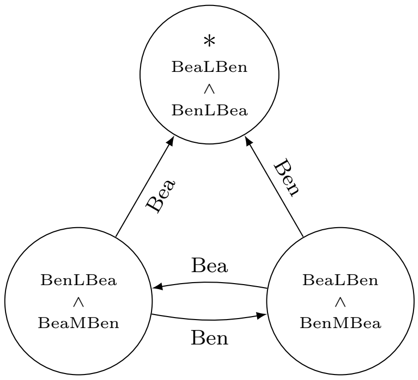

Figure 4: Plausibility belief model after the revelations from their friends about the other person's love.

Since $B_{Bea}BenLBea$ and $B_{Ben}BeaLBen$, these two beliefs will induce the following upgrades in the plausibility strategy model:
$\Uparrow_{Ben} BenLBea$ and $\Uparrow_{Bea} BeaLBen$. These two upgrades give the plausibility strategy model represented in Figure 5. And we have found another fix-point between the two models!

$$\begin{tikzcd}
(0,0)^* & (1,0) \arrow[l, "Beatrice"'] \\
(0,1) \arrow[u, "Benedick"] & (0,0) \arrow[l, "Beatrice"'] \arrow[u, "Benedick"']
\end{tikzcd}$$

Figure 5: Table representing the strategy game of Benedick and Beatrice after the love upgrades induced by their new beliefs.

6 Conclusions and Future Work

In this project we have seen the IERS characterization introduced in Cui et al. and the corrected solution for the Traveler's Dilemma. After that we have given IERS a plausibility characterization, we have compared this characterization to a preexisting one and we have seen an example of iterated game. The plausibility characterization allows to study both repeated games

and one shot games, thus it is more general and convenient to use than the PAL characterization.
As we have seen, through the IERS algorithm we find a fix-point model as our game solution. We saw two different representations of this procedure using modal logic and it could be interesting to see how modal $\mu-$calculus can help us find a synthetic formula for describing IERS.
For both the two previous modal languages we have the following result:

Theorem 25. *The game solution given from the IERS is the same as the limit set of worlds for repeated announcement of Ra^{re} ($= \bigwedge_{i\in N} Ra_i^{re}$), which is defined inside the full game model by*

$$\nu x. \bigwedge_{i\in N} Ra_i^{re} \wedge x$$

.

Proof. The first statement of the theorem follows from Theorem 14 for the PAL characterization (Theorem 23 for the Radical Upgrade characterization). Let's see the second statement.
By the definition of greatest fix-point, any world in the set P defined by $\nu x. \bigwedge_{i\in N} Ra_i^{re} \wedge x$ satisfies $\bigwedge_{i\in N} Ra_i^{re} \wedge x$. Thus the formula $\bigwedge_{i\in N} Ra_i^{re}$, being a logical consequence of this, also holds throughout P, and a further public announcement of rationality ($\Uparrow Ra^{re}$) has no effect.
On the other hand, the announcement limit for $\bigwedge_{i\in N} Ra_i^{re}$ is by definition a subset P of the current model that is contained in the set $\bigwedge_{i\in N} Ra_i^{re} \wedge x$. Thus, it is contained in the greatest fix-point for the monotonic operator matching this formula. □

Concerning further work first it would be interesting to see if it is possible to prove the completeness theorem for both logics presented in this work. The plausibility model might also be thought for other algorithms and it might be given a more general version of Theorem 23. Finally, concerning the example of Benedick and Beatrice, it would be nice to study how to formalize the relation between the plausibility strategy model and the plausibility belief model.

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